Math Reasoning, Practice Final Exam.

May 9, 2013

There are 8 questions worth 10 points each. Please write clearly and give complete proofs.

Problem 1.
Find all solutions $(x, y) \in \mathbb{Z}^2$ to the equation

$$5x + 6y = 13.$$

Problem 2.
Find all solutions $(x, y) \in \mathbb{Z}_8^2$ to the equation

$$5x + 6y = 13 \bmod 8.$$

Problem 3.
Find all integers $x \in \mathbb{Z}$ satisfying the following.

$$x \equiv 1 \bmod 13; \; x \equiv 5 \bmod 23; \; x \equiv 9 \bmod 15.$$

Problem 4.
Let a and b be relatively prime.
Show that the integers

$$0, a, 2a, \ldots, (b-1)a$$

represent all elements in $\mathbb{Z}_b$.

Problem 5.
Find all rational solutions $x \in \mathbb{Q}$ to the equation

$$9x^3 - 28x + 9 = 0.$$

Problem 6**.**
Prove that the sum

$$\sum_{k=1}^{n} \frac{1}{k^3} = \left(\frac{n(n+1)}{2}\right)^2.$$

Problem 7.
(a) Show that if 7 divides $10x$ then 7 divides x.

(b) Given an integer x form a smaller one y by taking the number whose decimal expansion is the one for x with the last digit removed, and then subtracting twice this last digit. For example, if $x = 12345$ then we take $y = 1234 - 2 \cdot 5 = 1224$.

Show that x is divisible by 7 if and only if y is divisible by 7.

Problem 8**.**

Let $< a >= \{a_n\}$ be a Cauchy sequence and $< b >= \{b_n\}$ be another sequence.

(a) If $|b_n| < 1$ for all n and $< a >\to 0$, show that the sequence $< ab >= \{a_n b_n\}$ is Cauchy.

(b) Show that the conclusion of part (a) may be false if $|b_n| < 1$ but $< a >$ does not converge to 0.