Existence and Stability of Foliations by J–Holomorphic Spheres

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September 22, 2009

Abstract

We study the existence and stability of holomorphic foliations in dimension greater than 4 under perturbations of the underlying almost complex structure. An example is given to show that, unlike in dimension 4, J–holomorphic foliations are not stable under large perturbations of almost complex structure.

1 Introduction

The theory of pseudoholomorphic curves was introduced in Gromov's seminal paper [Gro85]. There is a Fredholm theory showing that, for generic almost-complex structures J, pseudoholomorphic, or J-holomorphic, curves appear in finite dimensional families, with the dimension given by the Riemann-Roch theorem. Furthermore, in the presence of a taming symplectic form, suitable moduli spaces of J-holomorphic curves are compact modulo bubbling. These results have many important applications in symplectic topology. Notably they lead to Gromov-Witten invariants and Floer homology, which have been the main methods for establishing rigidity results in symplectic and contact topology.

Applied to symplectic manifolds of dimension 4 the theory of pseudoholomorphic curves is especially powerful and it becomes possible to prove classification results which are as yet inaccessible in higher dimensions. For example, symplectic forms on $S^2 \times S^2$ are classified, see [Tau96] and [Gro85], their symplectomorphism groups are well understood, see [Gro85] and [AM00], and the Lagrangian spheres are all known to be symplectically equivalent, see [Hin04]. These results all rely on the existence of foliations by J-holomorphic spheres. More precisely, they utilize the following theorem of Gromov. We say that a homology class $A \in H_2(X)$ is ω–minimal if $\omega(A) = \min_{B \in H_2(X), \omega(B)>0} \omega(B)$.

Theorem 1.1. *Let (X, ω) be a symplectic 4-manifold with a tamed almost-complex structure J and suppose that there exists an embedded symplectic sphere in a homology class A satisfying $A \bullet A = 0$ and A is ω–minimal.*

Then X is foliated by the images of J-holomorphic spheres homologous to A. The foliations vary smoothly with the almost-complex structure J.

The aim of paper is to investigate the extent to which this remains true when X has higher dimension.

It turns out that in general Theorem 1.1 is false if X is allowed to have dimension greater than 4. The existence of J-holomorphic spheres in the class A can be guaranteed at least for an open set of almost-complex structures by imposing an index constraint. But even if a foliation is known to exist for a particular J, it is unstable in the sense that varying J, even in the most generic fashion, can cause the foliation to degenerate.

To be more precise, we recall that given a family of tame almost-complex structures J_t, $0 \leq t \leq 1$, on the symplectic manifold X^{2n} we can define the universal moduli space

$$\mathcal{M} = \{(u,t) | u : S^2 \to X \ J_t\text{−holomorphic}, [u(S^2)] = A\}. \tag{1}$$

Suppose that $c_1(A) = 2$, then $\mathcal{M}$ has virtual dimension $2n+5$ and if the family $\{J_t\}$ is regular then $\mathcal{M}$ is a manifold (with boundary) of dimension $2n+5$. Furthermore, in the generic case the projection map $T : \mathcal{M} \to [0,1]$, $(u,t) \mapsto t$ is a Morse function and for all but finitely many t the fiber $\mathcal{M}_t$ is a manifold of dimension $2n+4$ consisting of the J_t-holomorphic spheres in the class A. For such regular t there is a smooth evaluation map

$$e_t : \mathcal{M}_t \times_G S^2 \to X$$

where the equivalence relation G is reparameterization of the holomorphic spheres. Both $\mathcal{M}_t \times_G S^2$ and X are smooth $2n$-dimensional manifolds.

Definition 1.2. We say that X is *foliated* by J_t-holomorphic spheres in the class A if the map e_t, when restricted to some connected component of its domain, is a homeomorphism.

We say that X is *smoothly foliated* by J_t-holomorphic spheres in the class A if the map e_t, when restricted to some connected component of its domain, is a diffeomorphism.

When X is 4-dimensional, or when the almost-complex structure J_t is integrable, these two notions coincide, but in higher dimensions there exist foliations (at least if we allow nonregular curves) for which the corresponding evaluation map is a smooth homeomorphism that is not a diffeomorphism. An example is given in Remark 2.5.

The following result shows that Theorem 1.1 fails completely in dimension greater than four. Let (M, ω_M) be a symplectic manifold of dimension at least four.

Theorem 1.3. *There exists a regular family J_t of tame almost-complex structures on $(X, \omega) = (S^2 \times M, \sigma_0 \oplus \omega_M)$ such that $\mathcal{M}$ has a component $\mathcal{N}$ where the curves in $t^{-1}(0) \cap \mathcal{N}$ form a foliation of X but the curves in $t^{-1}(1) \cap \mathcal{N}$ do not, the curves are not disjoint.*

In fact, we can take J_0 to be a product structure on $S^2 \times M$ and so $\mathcal{M}_0$ has a single component consisting of curves with images $S^2 \times \{z\}$ for $z \in M$. Fixing a point $0 \in M$ we can further assume that the corresponding sphere $C_0 = S^2 \times \{0\}$ is J_t-holomorphic and regular for all t. However there exists a two parameter family of curves C_r in $\mathcal{M}_1$ which includes C_0 but with $C_r \cap C_0 \neq \emptyset$ for all r.

An analog of Theorem 1.1 does remain true if we impose restrictions on the J_t. In this paper we will explain how to guarantee the existence and stability of foliations in the case of integrable complex structures and additional restrictions on the curvature.

Theorem 1.4. *Let (X, ω, J) be Kähler with holomorphic bisectional curvature bounded from below by $c > -\pi/\omega(A)$, where $A \in H_2(X; \mathbb{Z})$ is an ω–minimal homology class with $GW^X_{0,1,A}(pt) = 1$. Then X is smoothly foliated by J–holomorphic spheres.*

We remark that if X is a product $(M, k\omega) \times (S^2, \sigma)$, where (M, ω, J) is Kähler and σ is the area form on S^2, then a product complex structure will satisfy the hypotheses of Theorem 1.4 whenever k is sufficiently large, and so will any other integrable complex structure that is sufficiently close to the product one.

Of central importance to the stability of foliations is the notion of superregularity as defined in [Don02].

Definition 1.5. A real–linear Cauchy–Riemann operator D on a complex vector bundle over S^2 is called *regular* if D is surjective. It is called *superregular* if $\ker D$ contains a collection of sections that are linearly independent over each point in S^2. A choice of such a collection of sections is called a *superregular basis* for D.

A J–holomorphic sphere u is called regular if the induced real–linear Cauchy–Riemann operator D_u on u^*TX is regular. An immersed J–holomorphic sphere u is called superregular if D_u acting on sections of the normal bundle is superregular.

Note that regularity does not imply superregularity and vice versa. For example, no regular linearized operator at a J–holomorphic curve in a 4–manifold with self-intersection number $\neq 0$ is superregular. We will give an example of a superregular operator that is not regular in Section 2.1.

One way to understand what it means for a regular linearized operator to be superregular is the following. Suppose u is regular and of index $2n + 4$, so that the moduli space of curves near u, modulo reparameterizations, is a smooth manifold of dimension $2n - 2$. Then, in the language of Section 3.4 in [MS04], the evaluation map from the moduli space of J–holomorphic curves near a map $u : S^2 \to X^{2n}$ is transverse to all $x \in \text{image}(u) \subset X$ if and only if u superregular. Thus in the superregular case the image of u locally forms part of a smooth foliation.

The paper is arranged as follows. We first establish the non-existence result Theorem 1.3 in Section 2. Then we discuss the integrable case in Section 3 to prove Theorem 1.4.

We thank the referee for a careful reading of the manuscript and some indispensable suggestions.

2 Non-stability of foliations

For clarity of exposition we will restrict ourselves to work in dimension 6. It is clear how to generalize this to higher dimension, e.g. by taking the product with another symplectic manifold with compatible almost complex structure. However, our construction does not work in dimension less than 6 since in that case Hirsch's theorem about immersions does not apply, and consequently Lemma 2.4 does not hold.

2.1 Superregular Operator with Cokernel

Here we will construct a superregular Cauchy-Riemann operator with nontrivial cokernel. This immediately gives examples of foliations by holomorphic spheres which are not smooth.

Throughout this section $N = S^2 \times \mathbb{R}^4$ denotes the trivial bundle. Let $\{\bar{e}_i\}_{i=1}^4$ be the standard basis of $\mathbb{R}^4$ and J_0 the standard complex structure. Using the trivialization of N we will frequently identify sections of N with functions from S^2 into $\mathbb{R}^4$. We let j be a complex structure on S^2.

We recall the structure of a real–linear Cauchy–Riemann operator

$$D : \Gamma(N) \to \Gamma(\Lambda^{0,1}T^*S^2 \otimes_{J_0} N)$$

acting on sections of N, where $\Lambda^{0,1}T^*S^2$ denotes the the bundle of (i,j)–antilinear complexified 1–forms on S^2, that is, the sub–bundle of the complexified cotangent bundle $T^*S^2 \otimes_{\mathbb{R}} \mathbb{C}$ consisting of elements η satisfying $i\eta = -\eta \circ j$, and the tensor product is taken over $\mathbb{C}$ identifying i with J_0, so elements of $\Lambda^{0,1}T^*S^2 \otimes N$ are sums of elements of the form $\eta \otimes \xi$, and we identify $i\eta \otimes \xi = \eta \otimes J_0\xi$.

The operator D acts on sections ξ of the complex vector bundle (N, J_0) with trivial connection ∇ via

$$D\xi = \frac{1}{2}\left(\nabla\xi + J_0\nabla\xi \circ j\right) + \frac{1}{2}Y\xi = \bar{\partial}_0\xi + \frac{1}{2}Y\xi$$

where $Y : N \to \Lambda^{0,1}T^*S^2 \otimes N$ is a real vector bundle homomorphism.

Recall Definition 1.5 of our use of the terms regular and superregular.

Lemma 2.1. *Let D be a superregular real–linear Cauchy–Riemann operator on (N, J_0) with superregular basis $\{e_i\}_{i=1}^4$. Let $\Phi : S^2 \times \mathbb{R}^4 \to N$ be the corresponding trivialization, that is, $\Phi(z,x) = \sum_{i=1}^n x_i\, e_i(z)$.*

Then for a function $f : S^2 \to \mathbb{R}^4$ we have $D(\Phi(z, f(z))) = \Phi_*\bar{\partial}_J f$, where $J : S^2 \to End(\mathbb{R}^4)$ is given by $J = \Phi^*J_0$ and $\bar{\partial}_J f = \frac{1}{2}\{df + J\, df \circ j\}$.

Proof. We write $f = (f_1, \ldots, f_4)$ and $J_0 e_i = \sum_k J_{i\,k} e_k$. Then

$$\begin{aligned}
D\Phi(z, f(z)) &= D\sum_{i=1}^{4} f_i(z)e_i(z) \\
&= \sum_{i=1}^{4} \bar{\partial} f_i \otimes e_i \\
&= \frac{1}{2}\sum_{i=1}^{4}(df_i \otimes e_i) + \frac{1}{2}\sum_{i=1}^{4}(df_i \circ j \otimes J_0\, e_i) \\
&= \frac{1}{2}\Phi_*(df) + \frac{1}{2}\sum_{i,k=1}^{4}\{J_{ik} df_i \circ j \otimes\, e_k\} \\
&= \Phi_*(\frac{1}{2}df) + \sum_{k=1}^{4}\{\frac{1}{2}(J\,df)_k \circ j \otimes\, e_k\} \\
&= \Phi_*(\frac{1}{2}df) + \Phi_*(\frac{1}{2}J\,df \circ j) \\
&= \Phi_*(\bar{\partial}_J f).
\end{aligned}$$

The second equality follows since $\{e_i\}$ lie in the kernel of D.

□

We need the following elementary observation.

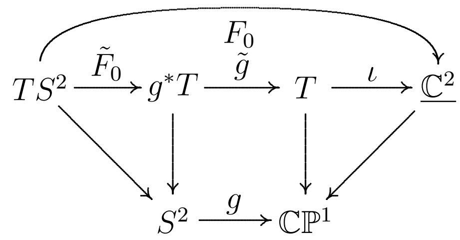

Figure 1: The construction of the map F_0.

Lemma 2.2. *Given any four sections $\{e_i\}_{i=1}^4$ of $N = S^2 \times \mathbb{C}^2$ that are linearly independent over each $z \in S^2$, there exists a unique real–linear Cauchy–Riemann operator $D = \bar\partial_0 + Y$, where $\bar\partial_0$ is the standard complex Cauchy–Riemann operator, $Y : N \to \Lambda^{0,1}T^*S^2 \otimes N$ is a vector bundle homomorphism, and $\{e_i\}_{i=1}^4$ is a superregular basis for D, i.e. so that $De_i = 0$ for $i = 1, \dots, 4$.*

Proof. Let $\nu_i = \bar\partial_0\, e_i$. Since the $\{e_i\}_{i=1}^4$ are linearly independent for all $z \in S^2$ we may define Y via $Y_z\, e_i(z) = -\nu_i(z)$. Thus $De_i = 0$ so $\{e_i\}_{i=1}^4$ is a superregular basis. Conversely, if $De_i = 0$ for $i = 1, \dots, 4$ then $Y_z(e_i(z)) = -\nu_i(z)$, defining Y uniquely. □

We now aim to construct a superregular real–linear Cauchy–Riemann operator on N which has a non–trivial cokernel. The following lemma clears some topological obstructions.

Lemma 2.3. *There exists a complex bundle monomorphism $F_0 : TS^2 \to \underline{\mathbb{C}^2}$.*

Here $\underline{\mathbb{C}^2}$ denotes the trivial $\mathbb{C}^2$–bundle over $S^2 = \mathbb{CP}^1$.

Proof. Consider the diagram in Figure 2.1. Let $T \to \mathbb{CP}^1$ be the tautological bundle, i.e.

$$T = \{(v, [z : w]| \, v \in \mathrm{span}_{\mathbb{C}}(z, w), \quad (z, w) \in \mathbb{C}^2\}$$

and let $g : S^2 \to \mathbb{CP}^1$ be a degree 2 map. Then TS^2 and g^*T have the same Chern class, so they are isomorphic complex vector bundles. Let $\tilde{F}_0 : TS^2 \to g^*T$ be a bundle isomorphism and let $\tilde{g} : g^*T \to T$ be the induced bundle homomorphism covering g.

Let $\iota : T \to \underline{\mathbb{C}^2}$ be the standard inclusion, i.e. $\iota(v, [z : w]) = (v, [z : w]) \in \mathbb{C}^2 \times \mathbb{CP}^1 = \underline{\mathbb{C}^2}$ and set

$$F_0 : TS^2 \to \underline{\mathbb{C}^2}, \qquad F_0 = \iota \circ \tilde{g} \circ \tilde{F}_0$$

F_0 is an injective complex vector bundle homomorphism because $\tilde{F}_0$, $\tilde{g}$ and ι are. □

Set $f_0 : S^2 \to \mathbb{C}^2$, $f_0(z) = 0$ and let $F_0 : TS^2 \to f_0^* T\mathbb{C}^2 = \underline{\mathbb{C}^2}$ as in Lemma 2.3. We aim to construct an actual immersion $f_1 : S^2 \to \mathbb{R}^4$ so that $(f_1, F_1 = df_1)$ has the same topological data as (f_0, F_0).

By Theorem 6.1 of [Hir59] (or alternatively by the h–principle) there exists an immersion $f_1 : S^2 \to \mathbb{R}^4$ with $F_1 = df_1 : TS^2 \to f_1^* T\mathbb{R}^4 = \underline{\mathbb{R}^4}$ together with a homotopy $f_t : S^2 \to \mathbb{R}^4$ connecting f_0 and f_1 covered by a homotopy of (real) monomorphisms $F_t : TS^2 \to \underline{\mathbb{R}^4}$ connecting F_0 and F_1. Here $\underline{\mathbb{R}^4}$ is again the trivial bundle and we implicitly made use of the standard (real) isomorphism $\underline{\mathbb{C}^2} = \underline{\mathbb{R}^4}$.

We need one more definition to construct a superregular operator with non–trivial cokernel. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^4$ and let $\mathcal{J}$ denote the space of complex structures on $\mathbb{R}^4$. Let $\mathcal{A}$ be the set of injective (real–linear) homomorphisms from one–dimensional complex vector spaces into $\mathbb{R}^4$. We define the map

$$\Phi : \mathcal{A} \to \mathcal{J} \tag{2}$$

in the following way. $A \in \mathcal{A}$ defines a splitting $\mathbb{R}^4 = V \oplus W$, where $V = \text{image}\, A$ and $W = V^\perp$. Define $J = \Phi(A)$ to be the unique $J \in \mathcal{J}$ that leaves this splitting invariant, makes A complex linear, and satisfies

$$\langle Je_1, e_2 \rangle = 1, \qquad \langle Je_1, e_1 \rangle = 0$$

on an oriented orthonormal basis e_1, e_2 of W.

Lemma 2.4. *There exists a superregular real–linear Cauchy–Riemann operator D with non–trivial cokernel.*

Proof. For each $z \in S^2$ define $J(z) = \Phi(F_1(z))$, where Φ is the map from Equation (2). Note that $\Phi(F_0(z)) = J_0$, so $\Phi(F_0)$ is covered by the map $G_0 = \text{Id} : S^2 \to GL(4, \mathbb{R})$, where the projection $\pi : GL(4, \mathbb{R}) \to \mathcal{J}$ is given by $\pi g = g^* J_0 = g^{-1} \circ J \circ g$. The map $\pi : GL(4, \mathbb{R}) \to \mathcal{J}$ is a bundle projection and thus has the homotopy lifting property. Let G_t be a lift of the homotopy $\Phi(F_t)$ to $GL(4, \mathbb{R})$.

$$\begin{array}{ccc} & & GL(4,\mathbb{R}) \\ & \overset{G_t}{\nearrow} & \downarrow \pi \\ S^2 \times I & \xrightarrow{\Phi(F_t)} & \mathcal{J} \end{array}$$

For $i = 1, \ldots, 4$ define sections $e_i(z) = G_1(z)\bar{e}_i$, where $\{\bar{e}_i\}_{i=1}^4$ denotes the standard basis of $\mathbb{R}^4$. By definition these are linearly independent for each $z \in S^2$, so by Lemma 2.2 we can choose Y so that these sections satisfy $D\, e_i = 0$, where $D = \bar{\partial}_0 + Y$. Thus D is superregular with superregular basis $\{e_i\}_{i=1}^4$.

Set $e_5(z) = G_1(z) f_1(z)$, and note that $\bar{\partial}_{J(z)} f_1 = 0$ by the definition of $J(z)$. So by Lemma 2.1 e_5 satisfies $D\, e_5 = 0$.

Thus the kernel of D is at least 5–dimensional, so D has non–trivial cokernel. □

Remark 2.5. Adding a suitable multiple of e_4 to e_5 we may assume that the pointwise inner product $\langle e_4(z), e_5(z) \rangle \geq 0$ but that the strict inequality does not hold. Then consider the map

$$e : \mathbb{R}^4 \times S^2 \to N,$$

$$(t_1, t_2, t_3, t_4, z) \mapsto \sum_{i=1}^{3} t_i e_i + t_4 e_5 + t_4^2 e_4.$$

This is clearly a smooth map giving a foliation of N by curves in the kernel of D, but its differential is not an isomorphism at $t_4 = 0$ wherever z satisfies $\langle e_4(z), e_5(z) \rangle = 0$. Using the almost complex structure J on N from Equation (7) we can regard these as J–holomorphic curves.

2.2 Family of Almost Complex Structures

Here we will apply the example from the previous section to construct a family D_s of Cauchy-Riemann operators for $s \in [-1, 1]$ such that D_s is regular for all s, superregular for s close to 2, but not superregular for s close to -1. The example can be globalized to produce the counterexample needed for Theorem 1.3.

Let $N \to S^2$ be a trivial complex rank 2 vector bundle with complex structure J_0 on the fibers. Fix an inner product $\langle \cdot, \cdot \rangle$ on N and let

$$D : \Gamma N \to \Gamma(\Lambda^{0,1} T^* S^2 \otimes N)$$

be a superregular real–linear Cauchy–Riemann operator with non–trivial cokernel as given by Lemma 2.4. By Riemann–Roch, D has index 4. Let $K = \ker(D)$ and fix a superregular basis $\{e_i\}_{i=1}^4$ and let $e_5 \in K$ be another section that is linearly independent from the $\{e_i\}_{i=1}^4$. Without loss of generality assume that e_5 is perpendicular to e_1, e_2, e_3 in L^2 and the pointwise inner product $h(z) = \langle e_4(z), e_5(z)\rangle \geq 0$ with $h(0) = 0$. Assume that $\|e_i\| = 1$, $i = 1, \ldots, 4$ and scale e_5 so that there exists $p_0 \in S^2$ with

$$h(p_0) = \langle e_4(p_0), e_5(p_0)\rangle = 1. \tag{3}$$

Let $C = \text{coker}(D)$ have dimension n and be spanned by the orthonormal basis $\{\eta_i\}_{i=1}^n \in C$. Let $\tilde{D} : K^\perp \to C^\perp$ be the restriction of D to $K^\perp$ and let $P : C^\perp \to K^\perp$ denote its inverse. Let

$$\pi_C : \Gamma(\Lambda^{0,1} T^* S^2 \otimes N) \to C$$

denote the orthogonal projection in L^2.

Lemma 2.6. *There exists a family of vector bundle homomorphisms* $Y_s : N \to \Lambda^{0,1} T^* S^2 \otimes N$, $s \in [-1, 1]$, *so that*

$$L_s = \pi_C \circ Y_s : \Gamma(N) \to C$$

is surjective and

$$K_s = \ker(L_s) \cap K = \mathit{span}\{e_1, e_2, e_3, s\, e_4 + (1 - s)e_5\}.$$

We will make use of the notation introduced above. Also recall the point $p_0 \in S^2$ from Equation (3), so the family of sections e_1^s defined via $e_i^s = e_i$ for $i = 1, \ldots, 3$ and $e_4^s = s\, e_4 + (1 - s)e_5$ are linearly independent vectors over p_0 for all $s \in [-1, 1]$. Let $U \subset S^2$ be an open neighborhood of p_0 so that the $\{e_i^s(z)\}_{i=1}^4$ still are linearly independent vectors over all $z \in \overline{U}$. Let $V \subset C^\perp$ denote the subspace of smooth sections in $C^\perp$ that are supported in U. Then we can define a family of homomorphisms Y_s from N to $\Lambda^{0,1} T^* S^2 \otimes N$ via functions $g_i^s : [-1, 1] \to V$ by

$$Y_s(e_i^s) = g_i^s.$$

Set $L_s = \pi_C \circ Y_s : \Gamma(N) \to C$ and $K_s = \ker(L_s)$. Note that by construction $e_i^s \in K_s$ for all $i = 1, \ldots 4$. We will prove Lemma 2.6 by finding suitable g_i^s so that L_s is surjective. Alternatively we can define a map

$$F : [-1, 1] \times V^4 \to [-1, 1] \times Hom(K, C), \qquad F(g_1^s, \ldots, g_4^s) = L_s|_K \tag{4}$$

and we need to show that the image of F contains a family of surjective homomorphisms. In order to prove this we need the following three lemmas.

Lemma 2.7. *Suppose that g_i^s are given so that $\dim(K_s) \leq m+4$ for some $0 \leq m \leq n$ and all $s \in [-1,1]$. Then there exists a continuous family of linearly independent sections $\{f_j^s\}_{j=1}^n \in K$ that are orthogonal to $\{e_i^s\}_{i=1}^4$ such that $\{f_j^s\}_{j=m+1}^n \in K_s^\perp \cap K$ for all s.*

Proof. Define $S_{m+4} \subset S_{m+3} \subset \ldots \subset S_4 = [-1,1]$ where $S_r = \{s \in [-1,1] | \dim(K_s) \geq r\}$. Then each S_r is a closed subset of $[-1,1]$. Restricted to S_{m+4} the vector spaces $K_s^\perp \cap K$ form a continuous vector bundle and so admit a continuous frame of $n-m$ sections $\{f_j^s\}_{j=m+1}^n$. More precisely, if $G(r,K)$ denotes the Grassmannian of r-dimensional subspaces of K, then the natural map $S_{r-1} \setminus S_r \to G(n+5-r,K)$ given by $s \mapsto K_s^\perp \cap K$ is continuous. Therefore by the Tietze extension theorem the map $S_{m+4} \to G(n-m,K)$ extends to $[-1,1]$ and so the corresponding vector bundle admits a continuous frame.

Arguing by induction, suppose that we have found continuous linearly independent sections $\{f_j^s\}_{j=m+1}^n$ of $K_s^\perp \cap K$ defined over S_r. By the extension theorem they extend as sections of K over $[-1,1]$. Let π_s denote the orthogonal projection from K onto $K_s^\perp$. The maps π_s vary continuously with s if we restrict to $S_{r-1} \setminus S_r$, and so $\{\pi_s f_j^s\}_{j=m+1}^n$ give continuous sections of $K_s^\perp \cap K$ over $S_{r-1} \setminus S_r$. Furthermore, these sections extend continuously the sections $\{f_j^s\}_{j=m+1}^n$ over S_r. This is because if $s \in S_r$ then $\pi_s f_j^s = f_j^s$ and the projection of $f_j^{s'}$ onto $K_{s'}$ remains small for s' close to s (as all sections in such $K_{s'}$ are close to sections in K_s). The $\{\pi_s f_j^s\}_{j=m+1}^n$ remain linearly independent on an open subset I of S_{r-1} containing S_r, in fact we may even assume that the sections are linearly independent on the closure $\overline{I}$. Now the natural map $S_{r-1} \setminus I \to G(n+5-r,K)$ extends to $[-1,1]$ and the sections $\{\pi_s f_j^s\}_{j=m+1}^n$ of the corresponding bundle restricted to $\overline{I} \setminus I$ therefore extend to continuous sections over $[-1,1]$, and in particular S_{r-1}, as before, using the Tietze extension theorem again. Thus we can conclude by induction to find continuous sections $\{f_j^s\}_{j=m+1}^n$ of $K_s^\perp \cap K$ over all of $[-1,1]$.

The remaining sections $f_j^s \in K$ for $j = 1, \ldots, m$ can then be constructed continuously over s and orthogonal to the e_i^s since the orthogonal complement of the sections already constructed forms a vector bundle over $[-1,1]$ which therefore admits a continuous frame field. □

Lemma 2.8. *Fix $s \in [0,1]$ and let $\hat{K} \subset K$ and $\hat{C} \subset C$ be non–empty vector spaces with $\hat{K}$ orthogonal to $\{e_k^s\}_{i=k}^4$. Then the map*

$$\hat{F} : V^4 \to Hom(\hat{K}, \hat{C}), \qquad \hat{F}(g_1, \ldots g_4) = \pi_{\hat{C}} \circ Y|_{\hat{K}},$$

where $Y : N \to \Lambda^{0,1}T^*S \otimes N$ is the homomorphism associated to the g_k, is nonzero.

Proof. Let $f \in \hat{K}$ and $\eta \in \hat{C}$. Let $a_k \in C^\infty(S^2, \mathbb{R})$ be such that $f = \sum_{k=1}^4 a_k e_k$ on U. Then $\hat{F}(g_1, \ldots g_4)(f) = \sum_{k=1}^4 g_k a_k$ on U.

If $\hat{F}$ was trivial, then the L^2 inner product

$$\sum_{k=1}^4 \langle g_k, a_k \cdot \eta \rangle_{L^2(U)} = \sum_{k=1}^4 \langle g_k \cdot a_k, \eta \rangle_{L^2(U)} = 0 \quad \forall g_k \in V \tag{5}$$

By choosing $g_j = 0$ for $j \neq k$ we conclude that each individual summand in the above formula vanishes. We claim that V (thought of as a space of sections over U) is dense in the space of L^2–sections ξ of $\Lambda^{0,1}T^*S^2 \otimes N$ restricted to U that satisfy

$$\langle \xi, \eta \rangle_{L^2(U)} = 0 \qquad \forall \eta \in C.$$

To see this, we construct a smooth, compactly supported L^2–approximation of any such ξ. Let $\|.\| = \|.\|_{L^2(U)}$ denote the L^2 norm of sections over U and fix $\frac{1}{2} > \varepsilon > 0$. Choose χ be a bump function supported in U so that $\|\chi\xi - \xi\| < \varepsilon/5$ and use mollifiers to define a smoothing ξ' of $\chi\xi$ that still has compact support in U and satisfies $\|\chi\xi - \xi'\| < \varepsilon/5$. In particular, $\|\xi - \xi'\| < 2\varepsilon/5$ and $\|\pi_C\xi'\| = \|\pi_C(\xi' - \xi)\| \leq \|\xi' - \xi\| \leq 2\varepsilon/5$, where π_C denotes the orthogonal projection from sections over U onto $C|_U$.

Given a bump function χ supported in U define the space C' as the image of $C|_U$ under multiplication with the bump function. For sharp enough bump function the orthogonal projection $\pi : C' \to C|_U$ is an isomorphism and $\|\pi - \mathrm{Id}\| \leq \varepsilon/2$, so $\|\pi^{-1} - \mathrm{Id}\| \leq \varepsilon$. Then define $\tilde{\xi} = \xi' - \pi^{-1}\pi_C\xi' \in V$, so

$$\|\tilde{\xi} - \xi\| \leq \|\xi' - \xi\| + \|\pi^{-1}\pi_C\xi'\| \leq 2\varepsilon/5 + (1+\varepsilon)2\varepsilon/5 \leq \varepsilon,$$

proving the claim.

Thus we conclude from Equation (5) that $a_k\eta|_U \in C|_U$. Recall that our Cauchy–Riemann operator $D = \bar{\partial}_0 + A$, where $A : N \to \Lambda^{0,1}TS^2 \otimes N$ is a real homomorphism. It's adjoint is given by $D^* = \bar{\partial}_0^* + A^*$, where $\bar{\partial}_0^* = -*\partial_0*$. For more details on Cauchy–Riemann operators see Appendix C.2 of [MS04]. Then over U for all k, as a_k are real valued,

$$0 = D^*(a_k\eta) = a_k\, D^*\eta - *(\partial a_k \wedge *\eta) = *(\partial a_k \wedge \eta \circ j).$$

Recall that the product of a (1,0)–form and a (0,1)–form is zero if and only if one of the factors is zero. The zeros of η are isolated since η is a non–trivial element in the cokernel of an elliptic operator. Thus, by continuity, $\partial a_k = 0$ everywhere in U.

The only real–valued antiholomorphic functions are constant, so the $\{a_k\}_{k=1}^4$ are constant functions over U. This means that f is a linear combination (over $\mathbb{R}$) of the e_k on U. By unique continuation, using that f and e_k are in the kernel of the operator D, we conclude that f is globally a linear combination of the e_k. But this contradicts the assumption that f is linearly independent of the e_k. □

Lemma 2.9. *Suppose that Y_s and corresponding sections f_j are given as in Lemma 2.7. Then there exists a smooth family $(g_i^s)_{i=1}^4 \in V^4$ such that for each of the corresponding maps L_s there exists an $f^s \in \mathrm{span}\{f_j^s\}_{j=1}^m$ such that $L_s(f^s)$ does not lie in the span of the $\{L_s(f_j)\}_{j>m}$.*

Proof. As in the previous lemma we express the $\{f_j^s\}_{j=1}^m$ as a linear combination of functional multiples of the $\{e_i^s\}_{i=1}^4$, at least over the open set U. Then we redefine the above linear map F from Equation (4) such that its range is paths of $m \times m$ matrices with entries determined by the L^2 inner products of the $\{L_s(f_j^s)\}_{j=1}^m$ with $\{\eta_j^s\}_{j=1}^m$ orthogonal to the $L_s(f_j)$ for all $j > m$. This linear map is nonzero for all s by Lemma 2.8. Therefore it has a positive codimensional kernel $M^s \subset V^4$ for all s and the lemma follows if we can find a continuous section of $M^\perp$ over $[0,1]$. But the rank of these vector spaces is again lower semicontinuous in s and so $M^\perp$ does indeed admit a section as above by first defining over points of minimal rank and then extending as before in the proof of Lemma 2.7. Finally we observe that since the $(g_i^s)_{i=1}^4 \in V^4$ satisfying the conclusions of the lemma form an open set, we may perturb and assume that our sections are in fact smooth. □

Proof of Lemma 2.6. We prove this by induction by perturbing g_i^s. Suppose that we have found g_i^s such that $\dim\ker(K_s) \leq m+4$ for all s and some $0 < m \leq n$. Then we can apply

Lemma 2.7 to find corresponding families of sections and thus a perturbation of the g_i^s using Lemma 2.9. If the g_i^s are chosen sufficiently small then the $L_s(f_j)$ are still linearly independent for all s and $j = m+1,..,n$. However for each s there is now an $f^s \in \mathrm{span}(\{f_j^s\}_{j=1}^m)$ with $L_s(f^s) \neq 0$ and independent of the $L_s(f_j)$ for $j > p$. Hence for each s we have that $\dim(K_s) \leq m + 4 - 1$ and the proof follows. □

Let Y_s and L_s be as in Lemma 2.6 and consider the family of real–linear Cauchy–Riemann operators

$$D_{s,t} = D + t\,Y_s. \tag{6}$$

The kernel of $D_{s,t}$ gets arbitrarily close to K_s as t gets small as described below. This result is well established in the literature (see e.g. [Kat95]), but we give a proof here for the convenience of the reader. In the following $\|\cdot\|$ denotes the L^2–norm.

Lemma 2.10. *There exists a constant $c > 0$ so that for all $0 < |t| < 1/2c$, $s \in [-1, 1]$ and $v_s \in K_s$, $D_{s,t}$ is surjective and there exists a unique $\xi_{s,t}(v_s) \in K_s^\perp$ so that*

$$v_s + \xi_{s,t}(v_s) \in \ker D_{s,t}.$$

Moreover $\|\xi_{s,t}(v_s)\| \leq 2tc\|v_s\|$.

Proof. Let $V_s = K_s^\perp \cap K$ denote the orthogonal complement of K_s in K and set

$$\tilde{L}_s = L_s|_{V_s} : V_s \to C.$$

Then $\tilde{L}_s$ is an isomorphism.

Let $W_s = K_s^\perp \cap \ker L_s \subset L^2(N)$ and consider the compact operator

$$F_s : \ker L_s \to W_s, \qquad F_s(\zeta) = \tilde{L}_s^{-1} \circ L_s(PY_s(\zeta)) - PY_s(\zeta),$$

where $P : C^\perp \to K^\perp$ is the inverse of $D|_{K^\perp} : K^\perp \to C^\perp$.

Note that $L_s \circ F_s = 0$ and $\tilde{L}^{-1}$ and P have image in $K_s^\perp$, so F_s is well defined. Let $c = \sup_{s\in[-1,1]} \|F_s\|_{L^2}$. For $|t|c \leq \frac{1}{2}$ and $v \in \ker L_s$ note that

$$\|\sum_{n=1}^{N} t^n F_s^n(v)\| \leq \sum_{n=1}^{N} |t|^n \|F_s^n(v)\| \leq \sum_{n=1}^{N} |t|^n c^n \|v\| < 2|t|c\|v\|$$

so we may define

$$\xi_{s,t}(v) = \sum_{n=1}^{\infty} t^n F_s^n(v),$$

also satisfying $\|\xi_{s,t}(v)\| \leq 2tc\|v\|$. Moreover,

$$D_{s,t} \circ F_s = D \circ F_s + tY_s \circ F_s = -Y_s + tY_s \circ F_s$$

and thus

$$\begin{aligned} D_{s,t}\left(v + \sum_{n=1}^{N} t^n F_s^n(v)\right) &= Dv + tY_s(v) + \sum_{n=1}^{N} \left(t^{n+1} Y_s F_s^n(v) - t^n Y_s F_s^{n-1}(v)\right) \\ &= Dv + t^{N+1} Y_s F_s^N(v), \end{aligned}$$

which converges strongly to Dv (in L^2) as $N \to \infty$, so

$$D_{s,t}(v + \xi_{s,t}(v)) = Dv, \qquad \forall v \in \ker L_s.$$

In particular $v + \xi_{s,t}(v) \in \ker D_{s,t}$ for all $v \in K_s$.

Next we show that $D_{s,t}$ is surjective. It suffices to show that the image of $D_{s,t}$ is dense as $D_{s,t}$ is Fredholm and thus has a closed image. By Hahn–Banach, it suffices to show that there does not exists $0 \neq \mu \in L^2(\Lambda^{0,1}T^*S^2 \otimes N)$ that annihilates the image of $D_{s,t}$. Suppose to the contrary such a μ exists. Write $\mu = \mu_0 + \mu_1$, where $\mu_0 \in C$ and $\mu_1 \in C^\perp$. Without loss of generality assume that $\mu_1 \neq 0$, otherwise, for $\zeta = \tilde{L}_s^{-1}(\mu_0) \in V_s$,

$$\langle D_{s,t}\zeta, \mu\rangle = \langle tY_s(\tilde{L}_s^{-1}(\mu_0)), \mu_0\rangle = t\langle \tilde{L}_s(\tilde{L}_s^{-1}(\mu_0)), \mu_0\rangle = t\|\mu_0\|^2 = t\|\mu\|^2 \neq 0.$$

Set $\zeta = P(\mu_1) - \tilde{L}_s^{-1} \circ L_s \circ P(\mu_1) \in \ker L_s$ and consider $\zeta + \xi_{s,t}(\zeta)$. Then

$$\langle D_{s,t}(\zeta + \xi_{s,t}(\zeta)), \mu\rangle = \langle D\zeta, \mu\rangle = \langle \mu_1, \mu\rangle = \|\mu_1\|^2 \neq 0.$$

This shows that $D_{s,t}$ is surjective.

The uniqueness of $\xi_{s,t}(v)$ satisfying $D_{s,t}(v+\xi_{s,t}(v))$ follows from the surjectivity of $D_{s,t}$. □

In particular the above Lemma guarantees that for any given $\delta > 0$ there exists $t_0 > 0$ so that for all $t < t_0$ the regular operators $D_{s,t}$ are superregular for $s \in [\delta, 1]$ and $D_{s,t}$ are not superregular for $s \in [-1, -\delta]$. To see this note that for s in that range, $e_4^s = s\, e_4 + (1 - s)e_5$ satisfies $\langle e_4(0), e_4^s(0)\rangle < 0$ and $\langle e_4(p_0), e_4^s(p_0)\rangle = 1$ by Equation (3). Thus near p_0 the tuple (e_1, e_2, e_3, e_4^s) forms an oriented basis of $\mathbb{R}^4$ and at the point 0 they form a basis with the opposite orientation. In particular there must be points in S^2 where the sections do not form a basis of $\mathbb{R}^4$. This remains true under small perturbations of the tuple (e_1, e_2, e_3, e_4^s).

A real–linear Cauchy Riemann operator D on N gives rise to an $\mathbb{R}$–invariant almost complex structure J on the total space of N, where $\mathbb{R}$ acts on N by scaling, in the following way. Choose a local complex trivialization $N = S^2 \times \mathbb{C}^2$ and write $D = \bar{\partial}_0 + \frac{1}{2}Y$, where $Y \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^2, \Lambda^{0,1}T^*S^2 \otimes \mathbb{C}^2)$. Utilizing the projections to each factor S^2 and $\mathbb{C}^2$ of N, referred to as the horizontal and vertical directions with complex structures j and i, respectively, we define the almost complex structure J at a point $x = (w, u) \in N$ acting on a vector $(h, v) \in T_xN$ via

$$J(h, v) = jh - iY(w, u)h + iv. \tag{7}$$

Note that J is independent of the trivialization chosen and indeed satisfies $J^2 = -\mathrm{Id}$. Moreover, if $f : (S^2, j_0) \to (N, J)$ with $f(z) = (w(z), u(z)) \in S^2 \times \mathbb{C}^2$ in the homology class of a section, then

$$\bar{\partial}_J f = \frac{1}{2}\{df + J\, df \circ j_0\} = \frac{1}{2}\{dw + j\, dw \circ j_0\} + \frac{1}{2}\left\{du + i\, du \circ j_0 + Y_{(w,u)}dw\right\}$$

Thus for $\bar{\partial}_J f = 0$ it is necessary that w is a holomorphic map from (S^2, j_0) to (S^2, j). But as f is homologous to a section this map has degree 1 and so is a biholomorphism. Thus, modulo this change of coordinates we may assume that $w(z) = z$ and $j = j_0$. In this case

$$\bar{\partial}_J f = \frac{1}{2}\left\{du + i\, du \circ j_0 + Y_{(w,u)}\right\} = D(u)$$

so maps $f : S^2 \to N$ in the class of a section are J–holomorphic if and only if they can be parametrized as a section $f(z) = (z, \xi(z))$ and $D\xi = 0$. Moreover note the the zero section is always a J–holomorphic section no matter what D is and that the linearization of $\bar{\partial}_J$ at the zero section is D.

Let ω be the canonical product symplectic form on N so that on each fiber it reduces to the Fubini-Study form and let $\tilde{J}$ be the canonical product complex structure on N and $\tilde{D}$ the associated Cauchy–Riemann operator. Given any symplectic 4-manifold (M, ω_M) there exists a symplectic embedding from U into $(X, \omega) = (S^2 \times M, \sigma_0 \oplus \omega_M)$ preserving the S^2 factors and mapping fibers tangent to M, where U is a suitable small neighborhood of the zero-section in N. Thus $\tilde{J}$ extends to a product complex structure on X which is tamed by ω, and X is smoothly foliated by regular $\tilde{J}$-holomorphic spheres.

Let D_s, $s \in [-1, 2]$ be a smooth family of real–linear Cauchy Riemann operators on N so that $D_s = D_{s,t}$ for some small fixed t and $s \in [-1, 1]$, where $D_{s,t}$ is the operator from Lemma 2.10, and D_s interpolates between D_1 and $\tilde{D}$ for $s \in [1, 2]$. Denote the associated family of almost complex structures by J_s. Note that the J_s are tamed by ω on a neighborhood U of the zero section in N. We now modify the family J_s to construct a family of almost complex structures $\tilde{J}_s$ on N with the property that $\tilde{J}_s = \tilde{J}$ outside of U and $\tilde{J}_s = J_s$ in an open neighborhood $V \subset U$ of the zero section so that $\tilde{J}_s$ is still tamed by ω. Using the above embedding we can then extend the family $\tilde{J}_s$ over X.

The family $\tilde{J}_s$ is tamed by the canonical symplectic structure on $X = S^2 \times M$, and $\tilde{J}_2$ is the product complex structure on X. Thus $\tilde{J}_2$ is regular (and superregular) and X is foliated by $\tilde{J}_2$–holomorphic spheres. By construction $\tilde{J}_s$ is regular for all curves outside of U, and it is regular for all curves contained in V for $s \in [-1, 1]$. Therefore, by possibly adding a small perturbation to the family $\tilde{J}_s$ for $s \in [1, 2]$ over $U \subset X$, and a perturbation for all $s \in [-1, 2]$ over $U \setminus V$ we may assume that the family $\tilde{J}_s$ is a regular family of almost complex structures and that $\tilde{J}_{-1}$ is regular. (The zero section in U may not necessarily be holomorphic for all $s \in [1, 2]$, but if not we can certainly compose the $\tilde{J}_s$ with a family of symplectomorphisms of X taking holomorphic curves onto this zero section, and so may assume this without loss of generality.)

Since $\tilde{J}_s = \tilde{J}$ outside of U, the complement of U is foliated by $\tilde{J}$–holomorphic spheres for all $s \in [-1, 2]$. But when $s = -1$ the linearized operator at the zero section is D_{-1}, which is not superregular by construction, so the foliation does not persist to a J_{-1}–holomorphic foliation of X.

This proves Theorem 1.3 in the case that $(X, \omega) = (S^2 \times M^4, \sigma_0 \times \omega_M)$.

3 Stability of Foliations for Integrable Complex Structures

In this section we show that holomorphic foliations are stable under perturbations of complex structure so that the holomorphic bisectional curvature remains bounded.

Definition 3.1. Let (M, J) be a complex manifold with a hermitian metric. The *holomorphic bisectional curvature* $H(p, p')$ of two J–invariant planes p and p' in T_xM is

$$H(p, p') = R(X, JX, Y, JY)$$

where R is the regular Riemannian curvature tensor and X and Y are unit vectors in p and p', respectively.

We say that (M, J) has holomorphic bisectional curvature bounded from above (below) by a constant c if $H(p, p') \leq c$ ($H(p, p') \geq c$) for all $x \in M$ and J–invariant planes $p, p' \subset T_xM$.

Recall that after choosing a local unitary frame of a hermitian vector bundle E over a complex manifold M we can consider the connection 1–forms, the connection matrix and the local curvature form $\Theta \in \Lambda^2(Hom(E, E))$. We write $\Theta \geq 0$ if the hermitian matrix

$$-i\langle\Theta_x; v; \bar{v}\rangle \in Hom(E_x, E_x)$$

is positive semi-definite for all $x \in M$ and holomorphic tangent vectors $v \in T_xM$. We say that two curvature forms relate as $\Phi \geq \Phi'$ if $\Phi - \Phi' \geq 0$. For details see page 79 of [GH78], from where we need the following result:

Lemma 3.2. *Let $G \to M$ be a holomorphic vector bundle of (complex) rank at least 2 over a complex manifold M with hermitian metric, and let $E \subset G$ be a holomorphic subbundle and $F = E^{\perp}$ the orthogonal complement of E in G. Then the local curvature form of F is greater than or equal to the curvature form of G restricted to F.*

Lemma 3.3. *Let (X, ω, J) be Kähler and let $u : S^2 \to X$ be a J–holomorphic sphere. Assume that the the holomorphic bisectional curvature of (X, J) is bounded from below by $c > \pi k/\omega[u]$. Then the pullback bundle u^*TX has no holomorphic line-subbundle with first Chern class less than or equal to k.*

Proof. Let F be a holomorphic line-subbundle of u^*TX, and let E be a complementary holomorphic subbundle, which exists by a result of Grothendieck [Gro57] if the real dimension of X is at least 4 and is taken to be empty otherwise. Let $E^{\perp}$ denote the orthogonal complement of E in u^*TX and denote the curvature of $E^{\perp}$ with respect to the connection induced by u^*TX by K. By Lemma 3.2 we know that $K(\cdot) \geq u^*H(\cdot, F)$ on any complex frame. Then

$$c_1(F) = c_1(E^{\perp}) = \frac{1}{2\pi}\int_{S^2} K \geq \frac{1}{2\pi}\int_{S^2} u^*H(\cdot, F) \geq \frac{1}{2\pi}c\int_{S^2} \|du\|^2 \mathrm{dvol} > k.$$

□

Recall Definition 1.5 for our use of the terms regular and superregular.

Lemma 3.4. *Let (X, ω, J) be Kähler so that the holomorphic bisectional curvature is bounded from below by $c > -2\pi/\omega(A)$, where $A \in H_2(X; \mathbb{Z})$. Then any J–holomorphic sphere in the class of A is regular.*

If furthermore u is immersed, $c_1(A) = 2$, and the holomorphic bisectional curvature is bounded from below by $c > -\pi/\omega(A)$, then u is also superregular.

Proof. Let $u : S^2 \to X$ be a J–holomorphic curve representing A. By Lemma 3.3, and using that $c > -2\pi/\omega(A)$, the pullback tangent bundle u^*TX does not have a holomorphic line-subbundle with first Chern class less than -1. Thus u is regular by Lemma 3.3.1 in [MS04].

If furthermore u is immersed and $c > -\pi/\omega(A)$, then every holomorphic line-subbundle of the normal bundle has first Chern class ≥ 0. Since u is immersed and $c_1(A) = 2$, the first Chern class of the normal bundle is 0. Thus any holomorphic line-subbundle of the holomorphic normal bundle has first Chern class 0 and has a superregular basis. Again by Grothendieck we know that the normal bundle splits as a sum of line bundles, giving rise to a superregular basis of the normal bundle, so u is superregular. □

Corollary 3.5. *Let J_I^c be the space of integrable compatible complex structures on (X^{2n}, ω) with holomorphic bisectional curvature bounded from below by $c > -2\pi/\omega(A)$, where $A \in H_2(X;\mathbb{Z})$ with $c_1(A) = 2$. Further assume that there exists a J_0–holomorphic foliation of X by spheres in the class of A for some $J_0 \in \mathcal{J}_I^c$.*

Then X is foliated by J–holomorphic spheres for any $J \in \mathcal{J}_I^c$ in the path–component of J_0.

Proof. Let $J = J_1 \in \mathcal{J}_I^c$ be connected to J_0 via a path $\{J_t\}_{t\in[0,1]}$ and let $\mathcal{M}$ denote the family space of $\{J_t\}_{t\in[0,1]}$–holomorphic spheres in the class of A as in Equation (1) and let

$$\mathcal{M}^1 = \mathcal{M} \times_G S^2$$

denote the component of the one–pointed moduli space, modulo automorphisms.

By Lemma 3.4 all curves in $\mathcal{M}^1$ are regular, so $\mathcal{M}^1$ is a smooth manifold (of dimension $2n+1$) and the projection onto the $[0,1]$–factor is a submersion.

Denote the connected component of $\mathcal{M}^1$ containing the initial J_0–holomorphic foliation by $\tilde{M}$ and let $\tilde{M}_s = \{(u,p,t) \in \tilde{M} | \, t = s\} \subset \mathcal{M}$. Again, by Lemma 3.4, all curves in $\mathcal{M}_s$ are regular, so $\mathcal{M}_s$ is a smooth manifold (of dimension $2n$). The evaluation map $ev_s : \mathcal{M}_s \to X$ is holomorphic with respect to the natural complex structure on $\mathcal{M}_s$. It has degree 1, since ev_0 is of degree 1 by assumption. Thus ev_s is a diffeomorphism for all $s \in [0,1]$ and $\mathcal{M}_s$ is a smooth foliation of X. □

Remark 3.6. Note that any J–holomorphic sphere u (for integrable J) that is part of a smooth foliation is automatically regular and superregular. Indeed, any line subbundle of the normal bundle of u has non-negative first Chern class, since it has holomorphic sections induced by nearby curves. Since the first Chern class of the normal bundle at a leaf of a foliation is trivial all linear subbundles must have first Chern class 0, so the curve is regular and superregular.

Under more stringent curvature assumptions, we can prove the existence of foliations given conditions on a Gromov-Witten invariant. This is the substance of Theorem 1.4 that we are now prepared to prove.

Proof of Theorem 1.4. Let $J \in \mathcal{J}_I^c$ and let $\mathcal{M}$ be the space of J–holomorphic spheres in the class A. $\mathcal{M}$ is non-empty since the GW count is non-zero. Let $u \in \mathcal{M}$. By Lemma 3.3 we know that all holomorphic linear subbundles of u^*TX have first Chern class greater than or equal to 0 and u is regular. We claim that u is immersed. If not, then the first Chern class of the generalized tangent bundle, which is generated by sections on the domain with poles at the locations of the non–immersion points of order up to their multiplicity, is at least 4. But by the dimension formula we know that $c_1(A) = 2$, so the (generalized) holomorphic normal bundle (the quotient of u^*TX by the generalized tangent bundle) would, again using Grothendieck, contain a linear subbundle with first Chern class less than 0 which is impossible. Thus u is superregular.

Every $u \in \mathcal{M}$ is immersed, regular and superregular, so the differential of the evaluation map is a linear isomorphism everywhere, hence ev is a local diffeomorphism. The contribution of a curve to the GW count of a point class is 1, since ev is transverse to any $x \in X$, the moduli space of curves is a manifold, and J is integrable. Thus the evaluation map $ev : \mathcal{M} \to X$ has degree one. But holomorphic maps of degree 1 are injective and so our evaluation map is therefore a (global) diffeomorphism, showing that X is foliated by embedded holomorphic spheres. □

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