Solutions to Homework 1.

January 29, 2013

Proof of Proposition 1.43**.**

(a). Suppose that $x + z = y + z$.

Adding the additive inverse $-z$ to both sides gives $(x + z) + (-z) = (y + z) + (-z)$.

Using axiom A1 this implies $x + (z + (-z)) = y + (z + (-z))$ and therefore $x + 0 = y + 0$.

By A3 we conclude that $x = y$ as required.

(b). For any x in the field we have

$$x + x.0 = x(0 + 1) = x.1 = x.$$

The first equality here is true by axiom DL, the second by axiom A3 and the last by M3.

By A3 again this means that $x + x.0 = x + 0$. Hence using part (a) we get $x.0 = 0$.

(c). For any x and y in the field we have

$$xy + (-x).y = (x + (-x)).y = 0.y = 0.$$

Now the first inequality uses axiom DL, the second is the definition of additive inverses and the last one uses part (b).

Therefore by the definition of the inverse $-(xy) = (-x).y$.

(d). Putting $x = 1$ in (c) gives $(-1).y = -(1.y) = -y$. This holds for all y in the field as required.

(e). We have

$$(-x)(-y) = ((-1)x)((-1)y) = ((-1)(-1))(xy)$$

where the first inequality uses part (d) and the second one applied M1 and M2.

Using part (d) again, $(-1)(-1) = -(-1) = 1$ as -1 is the inverse of 1.

Therefore $((-1)(-1))(xy) = 1.(xy) = xy$ by axiom M3. This is the equality we need.

(f). Suppose $xz = yz$ and $z \neq 0$.
As $z \neq 0$ there is a z^{-1} and multiplying by this we get $(xz).z^{-1} = (yz).z^{-1}$.
Using M1 and the definition of z^{-1} this implies $x.1 = y.1$.
Therefore by M3 $x = y$ as required.

(g). Suppose $xy = 0$.
By part (b) we have $xy = x.0$.
If $x \neq 0$ then by part (f) we have that $y = 0$.
Reversing the roles of x and y we also see that if $y \neq 0$ then $x = 0$.
In conclusion either x or y is 0.

Proof of Proposition 1.45**.**
(F1). Suppose that $x \leq y$.

Since $y = y + 0 = y + (z + (-z))$ we then have $x \leq y + (z + (-z))$ which means $y + (z + (-z)) + (-x) \geq 0$.

Using A1 and A2 we can rewrite this as $(y+z)-(x+z) \geq 0$ which means $y + z \geq x + z$ as required.

(F2). Suppose that $x \leq y$ and $0 \leq z$.

Then $y - x \geq 0$ and $z \geq 0$. We break the proof into two cases.

If $y - x = 0$ or $z = 0$ then $(y - x).z = 0$ by 1.43 part (b) and therefore $xz = yz$ using DL and 1.43 (c).

If neither $y - x$ nor z are 0 then $y - x > 0$ and $z > 0$. Then by P2 we have $(y - x)z > 0$ and therefore $yz > xz$ using DL and 1.43 (c) again.

In either case we have $yz \geq xz$ as required.

(F3). Suppose $y - x \geq 0$ and $v - u \geq 0$.
We break the proof into cases.
First case: $y - x > 0$ and $v - u > 0$.
Then $(y - x) + (v - u) > 0$ by P1.
Rearranging using A1 this implies $(y+v)-(x+u) > 0$ and so $y+v > x+u$.
Second case: $y = x$ and $v - u \geq 0$.
Now by (F1) we have $v + y \geq u + y$.
As $y = x$ this means $v + y \geq x + u$.

Third case: $y \geq x$ and $v = u$.
Now by (F1) $y + v \geq x + v$ and so as $v = u$ we have $y + v \geq x + u$ again.
As these cases cover all possibilities we are done.

(F4). Suppose $0 \leq x \leq y$ and $0 \leq u \leq v$.
As $v \geq 0$ by (F2) we have $xv \leq yv$.
As $x \geq 0$ by (F2) we also have $xu \leq xv$.
Putting the two inequalities together implies that $xu \leq yv$ as required.

Problem 1.25**.**

Let x, y and z be the ages of the three daughters in order from youngest to oldest. Then $xyz = 36$.

The divisors of 36 are 1, 2, 3, 6, 9, 12 and 18. We know that x and y must each be one of these numbers, and their product must also be a divisor so that $z = 36/(xy)$ is at least as big as x and y. There are not many possibilities. In particular if $x = 3$ and $y \geq 6$ then $z \leq 36/18 = 2$ which is too small. Thus if $x = 3$ then $y = 3$ and in all other options x is 1 or 2.

These are the possibilities.

$(1, 1, 36)$; $(1, 2, 18)$; $(1, 3, 12)$; $(1, 6, 6)$; $(2, 2, 9)$; $(2, 3, 6)$; $(3, 3, 4)$.

The sums of the ages in the seven possibilities are 38; 21; 16; 13; 13; 11; 10 respectively.

As we know that the sum is not enough to determine which age list we have, the ages must be either $(1, 6, 6)$ or $(2, 2, 9)$. To decide between these we use the final piece of information that the eldest daughter is asleep upstairs. A 9 year old would likely be at school or awake when a census taker visits, so we conclude that the ages are 1, 6 and 6.

Problem 1.40**.**

We will apply set identities (from class or exercise 1.41) starting with $(A \cup B) - (A \cap B)$.

By definition the set can also be written as $(A \cup B) \cap (A \cap B)^c$, where the superscript c denotes the complement.

Applying deMorgan's law, the complement of an intersection is a union of complements and so we can replace $(A \cap B)^c$ by $A^c \cup B^c$.

Next we distribute the intersection with $(A \cup B)$ over this union to get

$$(A \cup B) \cap (A \cap B)^c = ((A \cup B) \cap A^c) \cup (A \cup B) \cap B^c).$$

In each of the two brackets distribute the intersection over the union to rewrite the set as

$$((A \cap A^c) \cup (B \cap A^c)) \cup ((A \cap B^c) \cup (B \cap B^c)).$$

Finally we note that the intersection of any set with its complement is empty and so our set is equal to $(B \cap A^c) \cup (A \cap B^c)$ or $(B - A) \cup (A - B)$ as required.

Problem 1.55**.** Let F be a field with three elements 0, 1 and x.

Axioms A3 and M3 give the addition table for 0 and the multiplication table for 1. Proposition 1.43 (b) gives the multiplication table for 0.

Thus for the multiplication table we only need to calculate $x.x$. Now, x has a multiplicative inverse (since $x \neq 0$) and so there exists a $w \in F$ with $x.w = 1$. The part of the table we already know shows that w cannot be 0 or 1, hence $w = x$ and we can complete the multiplication table.

By Proposition 1.43 (a), $x + 1 = 1 + x = 0$ since it cannot be 1 or x by Proposition 1.43 (a) (this would imply that $x = 1$).

Proposition 1.43 (a) also implies that each column and row of the addition table consists of three different elements. This lets us fill out the table as shown below.

•	0	1	x
0	0	0	0
1	0	1	x
x	0	x	1

+	0	1	x
0	0	1	x
1	1	x	0
x	x	0	1