Solutions to Homework 10.

April 17, 2013

Problem 8.26**.**

The Pythagorean triples are triples of the form $(2rs, r^2 - s^2, r^2 + s^2)$ or $(r^2 - s^2, 2rs, r^2 + s^2)$ for $r, s \in \mathbb{Z}$.

We aim to show that every integer $n > 2$ appears in a Pythagorean triple which does not include 0. First suppose that n is even. Then we can write $n = 2k$ with $k > 1$ and n appears in a Pythagorean triple by taking $r = k$ and $s = 1$ as the $2rs$ term. We just need to make sure that neither $r^2 - s^2$ nor $r^2 + s^2$ are 0. This is clear since $r = k > 1 = s$.

Next suppose that n is odd. Then $n = 2k + 1$ for some $k > 1$. Using the hint $n = (k+1)^2 - k^2$ and so appears in a Pythagorean triple with $r = k+1$ and $s = k$ as the $r^2 - s^2$ term. We need to make sure that neither $2rs$ nor $r^2 + s^2$ are 0, but again this is clear since $r, s > 0$.

Problem 8.31**.**

$g(x) = \frac{1}{f(x)}$ also defines a function from $\mathbb{Q}^*$ to $\mathbb{Q}^*$. Also

$$g(x+y) = \frac{1}{f(x+y)} = \frac{f(x) + f(y)}{f(x)f(y)} = g(y) + g(x).$$

Iterating this formula we see that

$$g(nx) = g((n-1)x) + g(x) = \cdots = ng(x)$$

for any natural number n.

Suppose that $c = f(1)$. Then $g(1) = \frac{1}{c}$. For any natural number n the formula above implies that $ng(\frac{1}{n}) = g(1) = \frac{1}{c}$. Hence $g(\frac{1}{n}) = \frac{1}{cn}$. Using the formula again for natural numbers n and m we get $g(\frac{m}{n}) = \frac{m}{cn}$.

If $x = \frac{-m}{n}$ with $m, n \geq 1$ then

$$g(1 + \frac{m}{n}) + g(\frac{-m}{n}) = g(1) = \frac{1}{c}$$

and so $g(\frac{-m}{n}) = \frac{1}{c} - \frac{n+m}{cn} = \frac{-m}{cn}$.

In general $g(\frac{m}{n}) = \frac{m}{cn}$ or $g(x) = \frac{x}{c}$ for $c \in \mathbb{Q}^*$. Therefore $f(x) = \frac{1}{g(x)} = \frac{c}{x}$.

Problem *A.7.*

The sequence is defined recursively by $a_1 = 2$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$ for $n \geq 1$.

Suppose that $2 < a_n^2 < 4$. Then

$$a_{n+1} - a_n = \frac{1}{2}(a_n + \frac{2}{a_n}) - a_n = \frac{1}{2a_n}(2 - a_n^2) < 0$$

and

$$a_{n+1}^2 = \frac{1}{4a_n^2}(a_n^2 + 2)^2 > \frac{4^2}{4 \cdot 2} = 2.$$

Hence $a_{n+1} < a_n$ but still $a_{n+1}^2 > 2$. Arguing recursively we see that $\{a_n\}$ is a decreasing sequence with $a_n^2 > 2$ for all n.

Suppose that $< a >$ is not a Cauchy sequence. Then there exists a k such that for all N we can find $n > m > N$ with $|a_n - a_m| = a_m - a_n > \frac{1}{k}$. Therefore, as $< a >$ is decreasing, given any term a_m we can always find a later term at least $\frac{1}{k}$ smaller. Repeating this argument r times starting from a_1 we can find a term which is at most $2 - \frac{r}{k}$. Letting $r = k$ we find a term less than or equal to 1, which is a contradiction as all terms have $a_n^2 > 2$. Hence $< a >$ is a Cauchy sequence.

Finally suppose that $< a >$ has a limit $L \in \mathbb{Q}$. The recursion relation implies that $2a_n a_{n+1} = a_n^2 + 2$. It is easy to check that the left hand side has limit $2L^2$ and the right hand side converges to $L^2 + 2$ as $n \to \infty$. Hence $2L^2 = L^2 + 2$ or $L^2 = 2$. But this is a contradiction to Theorem 8.13.

Problem *A.8.*

The hardest thing to check here is that Cauchy sequences are closed under multiplication. This was done in class but here is the argument again.

Suppose that $< a >$ and $< b >$ are Cauchy sequences.

As $< a >$ is Cauchy there exists an N_1 so that if $n, m \geq N_1$ we have $|a_n - a_m| < 1$. Therefore, if $n \geq N_1$ using the triangle inequality we have

$|a_n| < 1 + |a_{N_1}|$. Setting $A = |a_{N_1}|$ we can say that $|a_n| < 1 + A$ whenever $n \geq N_1$.

Similarly there exists a $B > 0$ so that $|b_n| < 1 + B$ whenever $n \geq N_2$.

Let $k \in \mathbb{N}$. Then as $< a >$ is Cauchy there exists an N_3 so that $|a_n - a_m| < \frac{1}{2k(1+B)}$ whenever $n, m \geq N_3$.

Similarly as $< b >$ is Cauchy there exists an N_4 so that $|b_n - b_m| < \frac{1}{2k(1+A)}$ whenever $n, m \geq N_4$.

Suppose that $n, m \geq N = \max\{N_1, N_2, N_3, N_4\}$. We will check that $|a_n b_n - a_m b_m| < \frac{1}{k}$. As k was an arbitrary natural number this will show that $< ab >$ is Cauchy as required.

We compute

$$|a_n b_n - a_m b_m| = |a_n b_n - a_m b_n + a_m b_n - a_m b_m|$$

$$= |(a_n - a_m)b_n - a_m(b_n - b_m)| \leq |a_n - a_m||b_n| + |a_m||b_n - b_m|.$$

Using the fact that $n, m \geq N$ the final term is at most $\frac{1}{2k(1+B)} \cdot (1 + B) + \frac{1}{2k(1+A)} \cdot (1 + A)$ as required.

Problem *A.9***.**

The definition of a subsequence is on page 277.

Suppose then that $< a >$ is a Cauchy sequence and $< b >$ is a convergent subsequence. So the kth term in the $< b >$ sequence is a_{n_l} where $n_1 < n_2 < ...$ is an increasing sequence of natural numbers.

We know that $< b >$ converges to a limit L and aim to show that $< a >$ also converges to L. In other words, for a given k we need to find an N so that if $n \geq N$ then $|a_n - L| < \frac{1}{k}$.

Convergence of $< b >$ means that there exists an N_1 so that when $n_l \geq N_1$ we have $|a_{n_l} - L| < \frac{1}{2k}$. As $< a >$ is Cauchy we can find an N_2 so that if $n, m \geq N_2$ then $|a_n - a_m| < \frac{1}{2k}$.

Suppose that $n \geq N = \max\{N_1, N_2\}$. We can also choose an l so that $n_l \geq N \geq N_1$. Then

$$|a_n - L| = |(a_n - a_{n_l}) + (a_{n_l} - L)| \leq |a_n - a_{n_l}| + |a_{n_l} - L|$$

by the triangle inequality. But the first term is less than $\frac{1}{2k}$ since $n, n_l \geq N_2$ and the second term is also less than $\frac{1}{2k}$ since $n_l \geq N - 1$. Hence $|a_n - L| < \frac{1}{k}$ as required.