Solutions to Homework 11.

April 24, 2013

Problem $A.10$**.**

Let x and y be real numbers represented by rational Cauchy sequences $< a >$ and $< b >$ respectively.

Then by definition xy is represented by the Cauchy sequence $< ab >$ which has terms $\{a_n b_n\}$. Also yx is represented by the Cauchy sequence $< ba >$ with terms $\{b_n a_n\}$.

These two sequences are exactly the same as multiplication of rational numbers is commutative. Therefore they define the same equivalence class and so give the same real numbers. Hence $xy = yx$ and multiplication is commutative.

Exactly the same kind of argument shows that addition and multiplication are associative. For example, if another real number z is represented by a Cauchy sequence $< c >$ then $(x+y)+z$ is represented by a Cauchy sequence with terms $\{(a_n + b_n) + c_n\}$ and $x + (y + z)$ is represented by a Cauchy sequence with terms $\{a_n + (b_n + c_n)\}$. Again, these sequences are the same as rational addition is associative and so in particular they give the same real numbers and so $(x + y) + z = x + (y + z)$.

Problem $A.11$**.**

0 is represented by the Cauchy sequence $< 0 >= \{0, 0, \dots\}$. This is an identity element for addition because if $x \in \mathbb{R}$ is represented by $< a >$ with terms $\{a_n\}$ then $0 + x$ is represented by a Cauchy sequence with terms $\{0 + a_n\}$. But this is just $< a >$ and so $0 + x = x$.

Let $x \in \mathbb{R}$ be as above. Then $-x \in \mathbb{R}$ is represented by $< -a >$ and $x + (-x)$ is represented by the Cauchy sequence $< a + (-a) >=< 0 >= 0$.

If $x \neq 0$ then x^{-1} is represented by a Cauchy sequence $< b >$ with terms $\{b_n\}$ where $b_n = \frac{1}{a_n}$ for n sufficiently large. Then $x \cdot x^{-1}$ is represented by a Cauchy sequence $< c >$ with terms $\{c_n\}$ where $c_n = a_n b_n = 1$ when n is sufficiently large.

1 is represented by the Cauchy sequence $< 1 >= \{1, 1, \dots\}$. Therefore the sequence $< 1 - c >$ has all sufficiently large terms equal to 0. In particular the sequence converges to 0 and so the real number $x \cdot x^{-1}$ is equal to 1.

Finally, as 0 is the identity for addition, $1 > 0$ means simply that 1 is a positive real number. Let $N = 1$ and $k = 2$. Then for all $n \geq N$ the nth term in the sequence for $< 1 >$ is greater than $\frac{1}{k}$. Hence by the definition of order on the real numbers 1 is positive as required.

Problem *A*.12**.**
Let x and y be real numbers represented by rational Cauchy sequences $< a >$ and $< b >$ respectively. Suppose that $x > 0$ and $y > 0$. Then there exist natural numbers N_1, k_1, N_2, k_2 so that if $n \geq N_1$ then $a_n \geq \frac{1}{k_1}$ and if $n \geq N_2$ then $b_n \geq \frac{1}{k_2}$.

We'll show that $xy > 0$. The real number xy is represented by a Cauchy sequence with terms $\{a_n b_n\}$. Set $N = \max\{N_1, N_2\}$ and $k = k_1 k_2$. Then if $n \geq N$ we have $n \geq N_1$ and $n \geq N_2$ and so $a_n b_n \geq \frac{1}{k_1}\frac{1}{k_2} = \frac{1}{k}$. Hence $xy > 0$ as required.

Problem *A*.14**.**
Let $n > m$. Then

$$|a_n - a_m| = |(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_{m+1} - a_m)|$$
$$< \sum_{k=m}^{k=n-1} |a_{k+1} - a_k| \leq \sum_{k=m}^{k=n-1} \frac{M}{2^k}$$

where the first inequality uses the triangle inequality and the second uses the hypothesis of the question.

Since $\sum_{k=m}^{k=n-1} \frac{M}{2^k}$ is a geometric series with $n - m$ terms having first term $\frac{M}{2^m}$ and ratio $\frac{1}{2}$ its sum is

$$\frac{\frac{M}{2^m}(1 - (\frac{1}{2})^{n-m})}{1 - \frac{1}{2}} = 2M(\frac{1}{2^m} - \frac{1}{2^m}) < \frac{M}{2^{m-1}}.$$

To show that $< a >$ is a Cauchy sequence suppose that we are given a natural number k. Then we can choose a natural number N with $2^{N-1} > Mk$.

Suppose that $n > m \geq N$. Then from the above

$$|a_n - a_m| \leq \sum_{k=m}^{k=n-1} \frac{M}{2^k} < \frac{M}{2^{m-1}} < \frac{M}{2^{N-1}} < \frac{1}{k}$$

since $m \geq N$ and $2^{N-1} > Mk$.

As we have found an N corresponding to any k the sequence $< a >$ is a Cauchy sequence as required.