Solutions to Homework 2.

February 6, 2013

Problem 2.3**.**

We assume that the assertion is supposed to hold for all real numbers. Then the statement is

$$(\forall a \in \mathbb{R})(\forall x \in \mathbb{R})(P(a,x) \Rightarrow Q(x)).$$

The negation of this is

$$(\exists a, x \in \mathbb{R})(P(a,x) \wedge \neg Q(x)).$$

This negation is true, indeed, set $a = 0$ and $x = 1$.

The following is true though

$$(\forall a \in \mathbb{R} \setminus \{0\})(\forall x \in \mathbb{R})(P(a,x) \Rightarrow Q(x)).$$

Problem 2.4**.**

(a) There exists an $x \in A$ such that for all $b \in B$ we have $b \leq x$.

(b) If $x \in A$ then there exists a $b \in B$ with $x \geq b$.

(c) There exist $x, y \in \mathbb{R}$ such that $f(x) = f(y)$ but $x \neq y$. (In other words, the function f is NOT injective.)

(d) There exists a $b \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f(x) \neq b$. (This is the same as saying that the image of f is not all real numbers.)

(e) There exists $x, y, \epsilon \in \mathbb{R}$ such that for all $\delta \in P$ we have $|x-y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$. (This statement is actually always false, so the original is always true.)

(f) There exists an $\epsilon \in P$ such that for all $\delta \in P$ there exist $x, y \in \mathbb{R}$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$. (This is the statement that f is NOT uniformly continuous, a notion from analysis we'll see later.)

Problem 2.19**.**

Let $P(x,t)$ be the statement that you can fool person x and time t. Then Lincoln's statement is

$$((\forall x)(\exists t)P) \wedge ((\exists x)(\forall t)P) \wedge \neg((\forall x)(\forall t)P).$$

The negation is

$$((\exists x)(\forall t)\neg P) \vee ((\forall x)(\exists t)\neg P) \vee ((\forall x)(\forall t)P).$$

In English this says that either there's a person who can never be fooled, or all people have a certain time when they can't be fooled, or everyone can be fooled all of the time. If all three of these seem unlikely then Lincoln is probably right.

Problem 2.21**.**

The negation can be written as "there exists an $n > 0$ such that for all real numbers x we have $x \geq 1/n$."

The negation is clearly false as for any n we just have to choose a real number x less than $1/n$.

Problem 2.44**.**

These statements can be checked using truth tables. For example this is part (a).

P	Q	$\neg Q$	$P \wedge \neg Q$	$(P \wedge \neg Q) \Rightarrow P$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

This means that the statement $(P \wedge \neg Q) \Rightarrow P$ is true whatever the truth values of P and Q.

Problem 2.47**.**

(a) We use direct proof. Suppose $P(x)$ is true. Then x is odd which means we can write $x = 2k+1$ for some $k \in \mathbb{Z}$. Therefore

$$x^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k = 4k(k+1).$$

We note that either k or $k+1$ is even and so $k(k+1)$ is divisible by 2. Therefore $4k(k+1)$ is divisible by 8 and $Q(x)$ is true.

(b) Here we use the contrapositive. Assume that $P(x)$ is false, which means that x is even. Then x^2 is even and $x^2 - 1$ is odd. Odd numbers are certainly not divisible by 8, so $Q(x)$ is false also and the proof is complete.

Problem 2.48**.**

(a) This is false. The negation is that there exists an integer x so that $P(x)$ is true but $Q(x)$ is false. We can take $x = 1$. Then x is odd and so P is true, but x is not twice an integer and so Q is false. Therefore the negation is true and hence the original statement is false.

(b) The statement to the left of the implication symbol is, in English, that for all integers x, x is odd. This is clearly false. By the definition of implies, a false statement implies anything. Therefore statement (b) is true!