Solutions to Homework 2.

February 12, 2013

Problem 2.35**.**

$$(x+1)^2 = (y+1)^2 \Leftrightarrow (x+1)^2 - (y+1)^2 = 0 \Leftrightarrow (x-y)(x+y+2) = 0.$$

If a product of two numbers is 0 then one of the two numbers themselves must be 0. We conclude that $(x+1)^2 = (y+1)^2$ if and only if $x - y = 0$ or $x + y + 2 = 0$. If we assume that x and y are distinct then $x - y \neq 0$ and so $x + y = -2$.

If we allow $x = y$ then the statement is

$$(x+1)^2 = (y+1)^2 \Leftrightarrow x = y \text{ or } x + y = -2.$$

Problem 2.38**.**

(a) This is true. xy is odd if and only if it is not divisible by 2. This occurs if and only if both x and y are not divisible by 2 and this means that x and y are odd.

(b) This is false. In particular the implication

$$xy \text{ even} \Rightarrow x, y \text{ both even}$$

is false. In context this statement is supposed to be universally quantified for x and y, therefore the negation is that

$$(\exists x, y)((xy \text{ even}) \wedge (x \text{ odd } \vee y \text{ odd})).$$

So to show that the original statement is false we need to show that this negation is true. For this, we can take $x = 1$ and $y = 2$. Then xy is even and x is odd.

Problem 2.54**.**

There are lots of possible arrangements of tokens in the circles but we can divide them into two types.

1. Arrangements where at least one circle contains an even number of white tokens.

2. Arrangements where every circle contains an odd number of white tokens.

Remember that 0 is thought of as an even number, so every arrangement is of type 1 or 2. Furthermore, our starting position is of type 1 and our target end position is of type 2. Therefore the conclusion will follow if we can prove the following lemma.

Lemma 0.1. *There is no move which takes us from an arrangement of type* 1 *to one of type* 2*.*

Proof. The move must be of type (a) or (b). If it is of type (a) then each circle contains an even number of tokens which are flipped. That is, one circle gets all tokens flipped and the other two will each contain two tokens which are flipped. Each flip changes the parity of the number of white tokens in a circle, so two flips will keep the parity the same. Therefore, if a circle starts with an even number of white tokens it will still have an even number after the move.

If the move is of type (b) then the circle we flip in will end with all tokens inside black, which means an even number of whites.

So in either case if we start with an arrangement of type 1 we will end with an arrangement of type 1 and the proof is complete. □

Problem 3.1**.**

For example $P(n) = (n < 100)$.

Problem 3.4**.**

We start by assuming $P(0)$ is true. Therefore according to the question with $n = 0$ either $P(0) \wedge P(1)$ or $P(0) \wedge P(-1)$ is true.

There exist examples where these statements are true but no others, and $P(n)$ has the property described.

Example. $P(n) = (n(n-1) = 0)$. Then $P(0)$ and $P(1)$ are true but all other $P(n)$ are false.

Checking the property there is nothing to show if $n \neq 0, 1$. If $n = 0$ then $P(n+1) = P(1)$ is also true. If $n = 1$ then $P(n-1) = P(0)$ is also true. So the property holds for these $P(n)$.

Problem 3.7**.**

We are interested in the statement $(\forall n \in \mathbb{N})(2n - 8 < n^2 - 8n + 17)$.

We will show that this is false, that is, the negation is true. The negation is $(\exists n \in \mathbb{N})(2n - 8 \geq n^2 - 8n + 17)$.

To show this, take $n = 5$. Then $2n - 8 = 2$ and $n^2 - 8n + 17 = 2$ and so the negation is true.

Problem 3.9**.**

We are interested in the statement $(\forall n \in \mathbb{N})(\frac{2n-18}{n^2-8n+8} < 1)$.

We will show that this is false, that is, the negation is true. The negation is $(\exists n \in \mathbb{N})(\frac{2n-18}{n^2-8n+8} \geq 1)$.

To show this, take $n = 4$. Then

$$\frac{2n - 18}{n^2 - 8n + 8} = \frac{-10}{16 - 32 + 8} = \frac{10}{8} > 1$$

and so the negation is true.