Solutions to Homework 4.

February 21, 2013

Problem 3.15**.**

Let $P(n)$ be the statement $\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}$.

It's easy to see that $P(1)$ is true, so to prove $P(n)$ for all n, using the principle of induction we just need to show that $P(n) \Rightarrow P(n+1)$.

Using direct proof assume that $P(n)$ is true. Then

$$\sum_{i=1}^{n+1}(-1)^i i^2 = (-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n+1)^2$$

$$= (-1)^{n+1}\frac{n+1}{2}(-n+2(n+1))$$

$$= (-1)^{n+1}\frac{(n+1)((n+1)+1)}{2}$$

and so $P(n+1)$ is also true as required.

Problem 3.16**.**

Let $P(n)$ be the statement $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

It's easy to see that $P(1)$ is true, so to prove $P(n)$ for all n, using the principle of induction we just need to show that $P(n) \Rightarrow P(n+1)$.

Using direct proof assume that $P(n)$ is true. Then

$$\sum_{i=1}^{n+1} i^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= \frac{(n+1)^2}{4}(n^2+4(n+1)) = \frac{(n+1)^2}{4}((n+1)+1)^2$$

and so $P(n+1)$ is also true as required.

Problem 3.32**.**
The formula is

$$\prod_{i=2}^{n}(1-\frac{(-1)^i}{i}) = \begin{cases} 1/2 & \text{if } n \text{ even} \\ \frac{n+1}{2n} & \text{if } n \text{ odd} \end{cases}$$

So let $P(n)$ be the statement that the formula is true for n.

It's easy to see that $P(1)$ is true, so to prove $P(n)$ for all n, using the principle of induction we just need to show that $P(n) \Rightarrow P(n+1)$.

Using direct proof assume that $P(n)$ is true. If n is even then

$$\prod_{i=2}^{n+1}(1-\frac{(-1)^i}{i}) = \frac{1}{2}(1+\frac{1}{n+1}) = \frac{(n+1)+1}{2(n+1)}$$

and so $P(n+1)$ is also true as $n+1$ is odd.

If n is odd then

$$\prod_{i=2}^{n+1}(1-\frac{(-1)^i}{i}) = \frac{n+1}{2n}(1-\frac{1}{n+1}) = \frac{1}{2}$$

and so $P(n+1)$ is also true as $n+1$ is even and we are done in all cases.

Problem 3.38**.**
Let $P(n)$ be the statement that the second player has a winning strategy if the game finishes at $4n$. So $P(250)$ is the statement we're really interested in, but we will prove by induction that the statement is true for all $n \in \mathbb{N}$.

$P(1)$ is true since whatever number the first player adds, the second player can bring the total to 4 and win.

For the induction step suppose that $P(n)$ is true and we have a game up to $4(n+1)$. If the first player adds k then the second player can add $4-k$ to bring the total to 4. Then we're left with a game up to $4(n+1)-4 = 4n$ which we already know the second player has a strategy to win. Therefore $P(n+1)$ is also true as required.

Problem 3.57**.**
Here we let $P(n)$ be the statement that $1 \leq a_n \leq 2$ and use strong induction to prove that $P(n)$ is true for all n.

$P(1)$ is true by definition, for the induction step assume that $P(n)$ is true for all $n < N$ and we will try to show that $P(N)$ must be true.

We know that

$$a_N = \frac{1}{2}(a_{N-1} + \frac{2}{a_{N-2}}).$$

To get a lower bound for a_N we plug in the least possible value for a_{N-1} and the maximum possible value for a_{N-2}. By the strong induction hypothesis these are 1 and 2 respectively and so we get

$$a_N \geq \frac{1}{2}(1 + \frac{2}{2}) = 1.$$

To get an upper bound for a_N we plug in the maximum possible value for a_{N-1} and the minimum possible value for a_{N-2}. By the strong induction hypothesis these are 2 and 1 respectively and so we get

$$a_N \leq \frac{1}{2}(2 + \frac{2}{1}) = 2.$$

We conclude that $1 \leq a_N \leq 2$ as required.