# Solutions to Homework 5.

March 4, 2013

**Problem** 8**.**

(a) Using the Euclidean algorithm check that $gcd(224, 126) = 14$ and $14 = 4 \cdot 224 - 7 \cdot 126$.

(b) Using the Euclidean algorithm check that $gcd(221, 299) = 13$ and $13 = 3 \cdot 299 - 4 \cdot 221$.

**Problem** 9**.**

(a) Using the Euclidean algorithm check that $gcd(17, 13) = 1$ and $1 = 4 \cdot 13 - 3 \cdot 17$.

Since the numbers are relatively prime the equation has a solution and we get one from the equation above by writing

$$200 = 200 \cdot 1 = (-600) \cdot 17 + 800 \cdot 13.$$

Now as shown in class the set of all solutions is given by

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} | x = -600 + 13k, y = 800 - 17k\}.$$

(b) Using the Euclidean algorithm check that $gcd(21, 15) = 3$ and $3 = 3 \cdot 15 - 2 \cdot 21$.

Since $93 = 3 \cdot 31$ is divisible by 3 the equation has a solution and we get one from the equation above by writing

$$93 = 31 \cdot 3 = (-62) \cdot 21 + 93 \cdot 15.$$

Now the set of all solutions is given by

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} | x = -62 + 5k, y = 93 - 7k\}.$$

(c) Using the Euclidean algorithm $gcd(60, 42) = 6$. But $104 = 17 \cdot 6 + 2$ is not divisible by 6. Therefore the equation has no solutions.

(c) Using the Euclidean algorithm $gcd(588, 231) = 21$ and $21 = 2 \cdot 588 - 5 \cdot 231$.

Since $63 = 3 \cdot 21$ is divisible by 21 the equation has a solution and we get one from the equation above by writing

$$63 = 3 \cdot 21 = 6 \cdot 588 - 15 \cdot 231.$$

Now the set of all solutions is given by

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} | x = 6 + 11k, y = -15 - 28k\}.$$

**Problem** 17.

By Proposition 6.13 adding multiples of one of the numbers to the other does not change the greatest common divisor.

Therefore

$$gcd(a + b, a - b) = gcd((a + b) + (a - b), a - b) = gcd(2a, a - b)$$

and

$$gcd(a+b, a-b) = gcd(a+b, (a-b)-(a+b)) = gcd(a+b, -2b) = gcd(a+b, 2b).$$

**Problem** 18.

Notice that if a prime number divides $a^2$ then it also divides $a$ (see Proposition 6.7). Therefore if $a$ and $b$ are relatively prime so are $a^2$ and $b^2$, so

$$gcd(a, b) = 1 \Rightarrow gcd(a^2, b^2) = 1.$$

However $gcd(a, 2b)$ may or may not be 1. For example $gcd(1, 1) = 1$ and $gcd(1, 2) = 1$, but $gcd(2, 1) = 1$ and $gcd(2, 2) = 2$.

**Problem** 23.

By Theorem 6.21 with $a = 3$ and $b = 2$ we see that $n$, $n + 2$, $n + 4$ have distinct remainders modulo 3. Hence exactly one of the numbers has remainder 0 and so is divisible by 3. A number divisible by 3 is not prime unless it is 3 itself.

We conclude that one of our numbers has to be 3 and so $n = 1$ or $n = 3$. If $n = 1$ the numbers are 1, 3, 5 which are not all prime as 1 is not prime. So the only possibility is that $n = 3$ and the numbers are 3, 5 and 7, which are all prime.

**Problem 26.**
Let $P(n)$ be the statement

$$((n-1)^3 + n^3 + (n+1)^3 \text{is divisible by } 9).$$

Putting $n = 1$, $(n-1)^3 + n^3 + (n+1)^3 = 1 + 2^3 = 9$ which is divisible by 3 and so $P(1)$ is true.

We will show that all $P(n)$ are true by induction. Using the principle of induction we now just need to show that $P(n) \Rightarrow P(n+1)$.

Using direct proof assume that $P(n)$ is true, that is, 9 divides $(n-1)^3 + n^3 + (n+1)^3$.

Then

$$((n+1)-1)^3 + (n+1)^3 + ((n+1)+1)^3 = n^3 + (n+1)^3 + ((n-1)+3)^3$$

$$= n^3 + (n+1)^3 + (n-1)^3 + 9(n-1)^2 + 27(n-1) + 27$$
$$= ((n-1)^3 + n^3 + (n+1)^3) + 9(n-1)^2 + 27(n-1) + 27$$

which is divisible by 9 as required.