Solutions to Homework 8.

April 3, 2013

Problem 8.13**.**

By definition of $\frac{c}{d} \leq \frac{a}{b}$ we have $ad - bc \geq 0$.

Therefore $ad - bc + ac - ac \geq 0$ or equivalently $a(d-c) + c(a-b) \geq 0$.

Now since $d \geq c$ and $c \geq a \geq 0$ we have $a(d-c) \leq c(d-c)$. Hence $c(d-c) + c(a-b) \geq 0$ or $c(d-c+a-b) \geq 0$.

Next since $c \geq 0$ we have $d - c + a - b \geq 0$ or $d - c \geq b - a$ as required.

For the second part of the question we give a counterexample to show that the inequality is no longer true. Put $a = 3$, $b = 2$, $c = 6$ and $d = 4$. Then $a \leq d < c$ but still $\frac{c}{d} = \frac{3}{2} = \frac{a}{b}$. But now $d - c = -2$ and $b - a = -1$ and so $b - a > d - c$.

Problem $A.6$**.**

We need to check the axioms on page 16 to show that $\mathbb{Q}$ is an ordered field with 0, 1, $+$, $\cdot$ as defined in class.

The axioms $A0$, $M0$, $A4$ and $M4$ were done in class, $A3$ and $M3$ follow directly from the definitions and $0 = \frac{0}{1}$ and $1 = \frac{1}{1}$. $M1$, $A2$ and $M2$ are easily checked so here we check $A1$ and DL.

Let $x = \frac{a_1}{b_1}$, $y = \frac{a_2}{b_2}$ and $z = \frac{a_3}{b_3}$ be three rational numbers.

For $A1$ we need to show that $(x+y)+z = x+(y+z)$.

By the definition of addition we have

$$(x+y)+z = \frac{a_1b_2 + a_2b_1}{b_1b_2} + \frac{a_3}{b_3}$$

$$= \frac{(a_1b_2 + a_2b_1)b_3 + a_3b_1b_2}{(b_1b_2)b_3}.$$

Also

$$(x+y)+z = \frac{a_1}{b_1} + \frac{a_2b_3 + a_3b_2}{b_2b_3}$$

$$= \frac{a_1b_2b_3 + (a_2b_3 + a_3b_2)b_1}{b_1(b_2b_3)}.$$

The two expressions have the same numerator since integer multiplication is associative and distributative over addition. They have the same denominator as integer multiplication is associative. Therefore they represent the same rational number and axiom $A1$ holds.

For DL we need to show that $x(y+z) = xy + xz$.

By the definition of addition and multiplication we have

$$x(y+z) = \frac{a_1}{b_1} \cdot \frac{a_2b_3 + a_3b_2}{b_2b_3}$$

$$= \frac{a_1(a_2b_3 + a_3b_2)}{b_1(b_2b_3)}.$$

On the other hand,

$$xy + xz = \frac{a_1a_2}{b_1b_2} + \frac{a_1a_3}{b_1b_3}$$

$$= \frac{a_1a_2b_1b_3 + a_1a_3b_1b_2}{b_1^2b_2b_3}.$$

We need to show that the two formulas above represent the same rational number. That is, they are related, or

$$(a_1(a_2b_3 + a_3b_2))b_1^2b_2b_3 = (a_1a_2b_1b_3 + a_1a_3b_1b_2)b_1b_2b_3.$$

But as integer multiplication is distributative over addition the left hand side is $a_1a_2b_1^2b_2b_3^2 + a_1a_3b_1^2b_2^2b_3$ and the right hand side is $a_1a_2b_1^2b_2b_3^2 + a_1a_3b_1^2b_2^2b_3$. These formulas are the same and so axiom DL holds too.

Finally, axioms $P1$, $P2$ and $P3$ for an ordered field were checked in class.