Solutions to Homework 9.

April 11, 2013

Problem 8.18.

We would like to show that if $x \in \mathbb{Q}$ and p is prime then $x^2 \neq p$.

We argue by contradiction and assume there exists an $x \in \mathbb{Q}$ with $x^2 = p$.

Let $x = \frac{r}{s}$ in lowest terms. Then $r^2 = ps^2$ and so $p|r^2$. As p is prime this means that $p|r$ and so we can write $r = kp$ for some $k \in \mathbb{Z}$. Then $k^2p^2 = ps^2$ and so dividing by p we get $k^2p = s^2$. Hence $p|s^2$ and again since p is prime $p|s$. This is a contradiction as $\frac{r}{s}$ was assumed to be in lowest terms and so r and s must be relatively prime.

Problem 8.20.

(a) We need to check that $f(x)$ has rational roots when $c = \pm 2$.

First let $c = 2$. Then $f(x) = x^6 + 2x + 1$ and so $f(-1) = 0$ as required.

Next, put $c = -2$. Then $f(x) = x^6 - 2x + 1$ and so we have $f(1) = 0$ and again we have a rational root.

(b) For a general integer c, suppose that x is a rational root, written as $x = \frac{r}{s}$ in lowest terms. By the Rational Zeros Theorem we have $r|1$ and $s|1$. Thus $r, s = \pm 1$ and $x = \pm 1$.

In the first case suppose $x = 1$. Then $f(1) = 1 + c + 1 = 0$ and so $c = -2$.

In the second case suppose $x = -1$. Then $f(-1) = 1 - c + 1 = 0$ and so $c = 2$.

Either way, we see that if $f(x)$ has a rational root then $c = \pm 2$, or in other words there are no rational roots when $c \neq \pm 2$.

Problem 8.21.

Suppose that $x = \frac{r}{s}$ is a rational root in lowest terms, which we remember means $s > 0$ and is as small as possible. By the Rational Zeros Theorem $r|2$

and $s|2$. Therefore the possibilities are $r = \pm 1, \pm 2$ and $s = 1, 2$. This gives the possibilities for x as $x = \pm 1, \pm 2, \pm\frac{1}{2}$.

As all terms in $f(x)$ have positive coefficients there are no positive roots, so in fact $x = -1, -2, -\frac{1}{2}$. Checking these, only $x = -1$ is a root. In conclusion there is exactly one rational root, namely -1.

As $x = -1$ is a root, $(x+1)$ is a factor and factorizing we can write $f(x) = (x+1)(2x^2 - x + 2)$. Note that $2x^2 - x + 2 = 2(x - \frac{1}{4})^2 + 2\frac{1}{8}$ and so is always positive. Hence we see that -1 is actually the only real root too.

Problem 8.22**.**

The kth root of an integer n is a solution to the equation $x^k - n = 0$.

If x is rational then we can write $x = \frac{r}{s}$ in lowest terms and by the Rational Zeros Theorem $s|1$. This means that x is an integer as required.

Problem 8.23**.**

The roots of $f(x) = ax^2 + bx + c$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If these numbers are rational then we can solve to see that $\sqrt{b^2 - 4ac}$ must be rational, and hence, by Theorem 8.14 for instance, $b^2 - 4ac$ is a perfect square, say n^2.

We know that b is odd. As $4ac$ is even this means $b^2 - 4ac = n^2$ is odd and so n is odd. Write $b = 2k + 1$ and $n = 2l + 1$ for some integers k and l.

Then

$$4k^2 + 4k + 1 - 4ac = 4l^2 + 4l + 1$$

which simplifies to

$$ac = k^2 - l^2 + k - l = (k - l)(k + l + 1).$$

Now $k - l$ and $k + l + 1$ differ by $2l + 1$, which is odd, and so exactly one of these numbers must be even. Therefore the product on the right hand side is even. Hence ac is even and so we conclude that a and c cannot both be odd as required.