Isotopies of high genus Lagrangian surfaces

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Abstract

It is shown that in a symplectic 4-manifold any two C^0 close, homotopic Lagrangian submanifolds are smoothly isotopic.

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1 Introduction

In this paper we address a local version of the isotopy problem for Lagrangian surfaces in a symplectic 4-manifold (M, ω). This question was first raised by V. Arnold in [1]. A Lagrangian submanifold L is one for which $\omega|_L$ vanishes. In general we would like to classify homotopic Lagrangian submanifolds up to smooth isotopy or better still Lagrangian isotopy, that is, smooth isotopy through Lagrangian submanifolds. Equivalence classes are called Lagrangian knots. Here we show that in a sufficiently small neighborhood of a given Lagrangian surface there are no Lagrangian knots up to smooth isotopy. More precisely our result can be stated as follows.

Theorem 1.1. *Let $T^*\Sigma$ be the cotangent bundle of a Riemann surface with its canonical symplectic structure and $L \subset T^*\Sigma$ be a connected Lagrangian submanifold homologous to Σ. Then L is smoothly isotopic to Σ.*

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In the case when Σ has genus 0 or 1 the above Theorem 1.1 is due to Y. Eliashberg and L. Polterovich, see [6]. In fact, work of the first author, see [13], shows that if Σ has genus 0 then all such Lagrangian spheres are Lagrangian isotopic to the zero-section. However we remark that it is not true in general that isotopic spheres are Lagrangian isotopic, see the work of P. Siedel, [26]. If Σ has genus 1 work of the second author [20] shows again that all such L are Lagrangian isotopic. The question of whether or not in higher genus cases all such isotopic Lagrangians are Lagrangian isotopic remains open.

In general symplectic 4-manifolds there exist homologous high genus Lagrangian submanifolds which are not smoothly isotopic, see the work of D. Park, M. Poddar and S. Vidussi, [24].

2 Proof of the theorem

In this section we prove Theorem 1.1. Since the result is known when Σ has genus 0 or 1 we will assume throughout that Σ has genus $g > 1$. Let σ be an area form on Σ of total area $2g - 2$. Let $\pi : T^*\Sigma \to \Sigma$ be the projection along the fibers. The cotangent bundle $T^*\Sigma$ carries a canonical symplectic form $\omega_0 = d(\lambda_0)$, where $\lambda_0 = pd\pi$ is the Liouville form. The zero section Σ is Lagrangian with respect to ω_0.

We can also think of $T^*\Sigma$ as a tubular neighborhood of a symplectic submanifold Σ. Then $T^*\Sigma$ carries another symplectic form τ which is symplectic on the fibers and such that $\tau|_\Sigma = \sigma$. Let $r : T^*\Sigma \to [0, \infty)$ be the length function with respect to an Hermitian metric on (the complex line bundle) $T^*\Sigma$. We denote the levels by $T^r\Sigma$. Then the unit circle bundle $\pi : T^1\Sigma \to \Sigma$ carries a connection α with $d\alpha = \pi^*\sigma$. We can arrange that $\tau|_{T^r\Sigma} = f(r)d\tilde{\alpha}$ where $\tilde{\alpha}$ is the pullback of the form α on $T^1\Sigma$ and f is decreasing towards 0 as r approaches ∞.

For ϵ sufficiently small, $\Omega_\epsilon = \omega_0 + \epsilon\tau$ is also a symplectic form on $T^*\Sigma$.

We reparameterize ω_0 such that outside of a large compact set $T^{\leq r_0}\Sigma$ it is given by $d(e^r\tilde{\lambda_0})$ where $\tilde{\lambda_0}$ denotes the pullback of the Liouville form from the unit tangent bundle. Also outside of $T^{\leq r_0}\Sigma$ we extend τ by extending the function f to a decreasing function $g(r)$ with $g = -e^r$ outside of a (larger) compact set. Then we define a new form ω on $T^*\Sigma$ by $\omega = \Omega_\epsilon$ on $T^{\leq r_0}\Sigma$ and $\omega = d(e^r\tilde{\lambda_0} + \epsilon g(r)\tilde{\alpha})$ elsewhere. We note that ω is a symplectic form for ϵ sufficiently small and that the fibers of $T^*\Sigma$ are ω-symplectic planes of infinte area.

Let V be a tubular neighborhood of our Lagrangian submanifold $L \subset (T^{\leq r_0}\Sigma, \omega_0)$.

Lemma 2.1. *There exists an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ the Lagrangian L can be isotoped to an Ω_ϵ symplectic surface within V.*

Proof This is a slight modification of Proposition 2.1.A in [6]. Let σ be a symplectic form on V such that $\sigma|_L$ is an area form of total area $2g-2$. Then $(\tau-\sigma)|_L$ is exact and so by the relative Poincaré Lemma there exists a 1-form λ on V such that $\sigma = \tau + d\lambda$. Let $\rho : V \to [0,1]$ have compact support and equal 1 close to L. Then there exists an ϵ_0 such that for $\epsilon \leq \epsilon_0$ the form $\Omega'_\epsilon = \omega_0 + \epsilon(\tau + d(\rho\lambda))$ is symplectic as is the linear family of forms connecting Ω'_ϵ to Ω_ϵ. As $\Omega'_\epsilon = \Omega_\epsilon$ away from V and L is Ω'_ϵ symplectic it follows from Moser's method that L can be isotoped to an Ω_ϵ symplectic surface inside V. □

The results from [14] imply that that all Ω_ϵ symplectic surfaces sufficiently close to Σ are isotopic to Σ. Thus we could conclude here if it were possible to arrange that the symplectic surface was contained in a suitably small symplectic neighborhood. However we could find no straightforward method of doing this. Instead we proceed as follows.

Lemma 2.2. *For ϵ sufficiently small, all connected symplectic surfaces S in $(T^*\Sigma, \omega)$ which are homologous to Σ and intersect the fiber over a point p exactly once transversally must be smoothly isotopic to Σ.*

Proof Let U be a neighborhood of p such that a given symplectic surface S intersects all fibers over points $q \in U$ transversally in a single point. By a small perturbation we may assume that $S \cap \Sigma$ is disjoint from $\pi^{-1}(U)$.

Let $h : \Sigma \to \mathbb{R}$ be a Morse function with a single minimum and all critical points contained in U. Then the gradient flowlines of h foliate the complement of the critical points of h by curves $\gamma(x) : (-\infty, \infty) \to \Sigma$ which lie in U for $|x|$ sufficiently large. Denote the critical points of h by $p_1, ..., p_N$.

Let $s_i \in S$ be the unique point with $\pi(s_i) = p_i$. Then we also assume that as subspaces of $T(T^*\Sigma)$ we have $T_{s_i}S = T(\pi^{-1}(p_i))^{\perp\Omega_\epsilon}$, the symplectic complement to the tangent space of the fiber.

Recall that for r sufficiently large $\omega|_{T^r\Sigma} = d\beta$ where $\beta = e^r\tilde{\lambda_0} + \epsilon g(r)\tilde{\alpha}$ is a contact form for ϵ sufficiently small. We observe that $\pi^{-1}(\gamma(\mathbb{R})) \cap T^r\Sigma$ is a cylinder C_γ foliated by the circles $F_x = \pi^{-1}(\gamma(x))$. Now, $\tilde{\lambda_0}$ vanishes on the F_x while $\tilde{\alpha}$ does not. Therefore $\ker \beta|_{C_\gamma}$ is a nonsingular line field l transverse to all F_x. In particular l has no closed orbits.

We claim that there exists an almost complex structure J_0 on $T^*\Sigma$ which is tamed by ω and satisfies the following properties. The surfaces $S \cap \pi^{-1}(U)$ and Σ are J_0 holomorphic; the contact planes $\ker\beta$ in $T^r\Sigma$ are J_0 holomorphic for some r sufficiently large; for all critical points p_i the disk $D_i = \pi^{-1}(p_i)$ is J_0-holomorphic.

The only requirement here which is not well known is the claim that it is possible to find a J_0 along $T^r\Sigma$ which simultaneously makes both the subbundles $\ker\beta$ and $\pi^{-1}(p_i)$ into J_0-holomorphic distributions. But the existence of such J_0 is established in a more general context by Theorem 7.4 in the article [4] of J. Coffey.

Let J_t, $0 \le t \le 1$ be a family of almost-complex structures on $T^*\Sigma$ coinciding with J_0 outside some $T^s\Sigma$, where $s < r$, and on $\pi^{-1}(U)$, such that S is J_1 holomorphic.

We next claim that for all t the cylinders C_γ can be foliated by circles which bound J_t holomorphic disks. These circles are transverse to l and at the ends of the cylinders the holomorphic disks converge to the fibers $\pi^{-1}(p_i)$ for p_i a critical point. The union of all disks over all cylinders gives a foliation of $T^{\le r}\Sigma$ by disks in the relative homotopy class of the fibers.

This claim follows from the theory of filling by holomorphic disks, see [5]. For each γ, the cylinder C_γ is foliated by the boundaries of embedded holomorphic disks near its ends. But as the cylinders are totally real the foliation extends to cover the whole cylinder. The only obstruction in this case is bubbling of holomorphic spheres inside $T^*\Sigma$ and bubbling of disks on the boundary. But as $\pi_2(\Sigma)$ is trivial such spheres do not exist. Bubbling of disks can be excluded as in [5] since all holomorphic disks with boundary on $T^r\Sigma$ must have boundary transverse to l. For embedded boundaries this fixes the homology class and prevents degeneracies.

The disks D_i constructed above are J_t holomorphic for all t and their intersection with S and Σ is transversal and in a single point. Therefore by positivity of intersections the same is true for all intersections of J_0 holomorphic disks with Σ and all J_1 holomorphic disks with S.

We fix a Riemannian metric on $T^*\Sigma$ which decays rapidly along the fibers. Then with respect to the restricted metric the centers of mass of our holomorphic disks give a smooth family of surfaces G_t. By the previous remark, it is clear that G_0 is smoothly isotopic to Σ

and G_1 is isotopic to S. □

Lemma 2.3. *The Lagrangian L can be isotoped to an ω symplectic surface in $T^*\Sigma$ intersecting the fiber over a point p transversally in a single point.*

This will follow from the following two lemmas.

First, let γ be a noncontractible curve in Σ and U be a neighborhood of $\pi^{-1}(\gamma)$. We can identify U with $D^2 \times S^1 \times (-s, s)$ where the D^2 factor has coordinates (x, y) and corresponds to the fibers of $T^*\Sigma$, the S^1 factor corresponds to γ and has coordinate θ and the interval has coordinate t.

We recall that we are working with a symplectic form $\omega = \omega_0 + \epsilon\tau$ where $\omega_0 = d(r\tilde{\lambda_0})$ is the canonical form on the cotangent bundle.

Lemma 2.4. *There exists a hypersurface $B \subset T^*\Sigma$ which is foliated by symplectic disks and contains a Lagrangian torus T intersecting each disk in a circle. The disks are symplectically isotopic to the fibers of $T^*\Sigma$. Furthermore there exists a symplectic isotopy of S to a surface intersecting B only on the interior I of T.*

Proof With the coordinates above on U, we may assume that $\omega_0 = dx \wedge dt + dy \wedge d\theta$ and $\tau = dx \wedge dy + d\theta \wedge dt$, at least close to the zero-section.

For any c a primitive of ω is given by

$$\lambda = cxdt - (1-c)tdx + yd\theta - \epsilon ydx - \epsilon td\theta.$$

The dual of λ with respect to ω is a conformally expanding vectorfield given by

$$Y = \frac{1}{1+\epsilon^2}(cx\frac{\partial}{\partial x} + ((1+\epsilon^2)y - c\epsilon t)\frac{\partial}{\partial y} + c\epsilon x\frac{\partial}{\partial \theta} + (1-c+\epsilon^2)t\frac{\partial}{\partial t}).$$

So, for instance letting $c = \frac{1}{2}$, we see that the corresponding contracting vectorfield $-Y$ vanishes only when $x = y = t = 0$, preserves $t = 0$, and retracts a (convex) neighborhood of $t = 0$ onto $x = y = t = 0$.

We can find similar contracting vectorfields defined near $t = \pm 2\epsilon^2$. We now follow a method of [13]. Let V be a convex neighborhood of, say, $t = 2\epsilon^2$. Near ∂V we can choose an almost-complex structure such that S is J-holomorphic, ∂V is J-convex and Y is the gradient of a J-convex exhaustion function f increasing to 0 near ∂V. We can perturb J further such that $df(Y)$ is constant near ∂V, and let χ be a decreasing function such that $\chi(0) = 0$ and $\chi(x) \equiv 1$ for $x < -\delta$, some small δ. Then the flow of $-\chi(f)Y$ now provides a compactly supported isotopy of S. The calculation in [13] shows that S remains symplectic during the isotopy, in particular near ∂V. Thus, repeating the procedure near $t = -2\epsilon^2$, we may assume that our surface S lies arbitrarily close, say within ϵ^6, to $x = y = 0$ when $t = \pm 2\epsilon^2$.

Now let $\phi(x, y)$ be a compactly supported function with $\phi(x, y) = \frac{y}{\epsilon}$ when $x^2+y^2 < \epsilon^6$ and $|d\phi| < \epsilon^2$. We can choose ϕ such that the surface $B = \{t = \phi(x, y)\}$ lies within $-2\epsilon^2 < t < \epsilon^2$. Further, we notice that

$$\omega(\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial t}, \frac{\partial}{\partial y} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial t}) = \epsilon + \frac{\partial \phi}{\partial y} > 0$$

and so B is foliated by symplectic planes, which coincide with the fibers away from the support of ϕ. We define the Lagrangian torus $T = \{x^2 + y^2 + t^2 = \epsilon^6\} \cap B$.

We now fix $c = \epsilon^2$ and study the flow of the corresponding vectorfield $-Y$ on $-2\epsilon^2 < t < 2\epsilon^2$, suitable cut-off near the boundary as before. Again, compact subsets contract towards $x = y = t = 0$. We claim that after following the flow for a sufficiently long time the image of S will intersect B only inside T. This is clear for points on S away from $-2\epsilon^2 < t < \epsilon^2$. For points starting near the boundary, the x and t coordinates strictly decrease and the rate of increase of the y coordinate is given by $-(1 + \epsilon^2)y + \epsilon^3 t$. So since $|t| < 2\epsilon^2$ the magnitude of the y coordinate can never exceed $2\epsilon^5$ (we recall that initially this magnitude is bounded by ϵ^6). Now, if a point on this flow intersects Σ then $x^2 + y^2 < \epsilon^6$ and so $t = \frac{y}{\epsilon}$ and $|t| < 2\epsilon^4$. Thus the intersection with B does indeed lie inside T as required and the lemma is complete.

□

Lemma 2.5. *There exists a symplectic disk asymptotic at its boundary to $\{\theta = 0, y = \epsilon t\}$ which intersects S in a single point and is isotopic to $\{\theta = 0, y = \epsilon t\}$ through symplectic disks disjoint from $B \setminus I$.*

Proof Let J_0 be an almost-complex structure on $T^*\Sigma$ tamed by ω and such that the fibers of $T^*\Sigma$ are J_0-holomorphic planes. It is easy to adjust J_0 such that $T^*\Sigma$ is still fibered by holomorphic planes coinciding with the cotangent fibers away from U but including the fibers of Σ. Then the $I \cap \{\theta = a\}$ for $a \in S^1$ are a family of J_0 holomorphic disks with boundary on the Lagrangian torus T.

If we degenerate J_0 along T then these disks can be thought of as finite energy planes in an almost-complex manifold $T^*\Sigma \setminus T$ with a concave cylindrical end isomorphic to $T^1T \times (-\infty, 0]$. Given a family of almost-complex structures J_t we can then study finite energy planes in $T^*\Sigma \setminus T$ asymptotic to a geodesic orbit in T^1T which projects in B to $0 \in S^1$. Such planes form a well-defined moduli space and since no other homology classes of geodesics in T can bound finite energy planes the moduli space is compact. If the J_t are all equal to J_0 on $B \setminus I$ then the planes avoid $B \setminus I$ by positivity of intersection, and the moduli space consists of a single plane when $t = 0$, again by positivity of intersection. Thus the space of planes is nonempty for all J_t. If we choose J_1 such that S is J_1-holomorphic then a J_1 holomorphic plane provides a disk satisfying the requirements of the lemma.

$\square$

Putting all of these isotopies together gives an isotopy of S as required for Lemma 2.3 and thus establishes our main theorem.

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