Note: Vectors appear in bold-face, and random entities will be capitalized. Notation should be equivalent to what we’ve used in class; please ask about any notational conventions which aren’t clear to you.

Simplify your answers as much as possible for full credit. If it is helpful to have the numerical value for an expression and you don’t have a calculator, you may make a reasonable estimate (for example, \( \sqrt{3} \approx 1.73 \)) and continue. You will not be penalized, though, for not evaluating expressions in your final answer which require a calculator or tables which have not been supplied.

| Problem 1 (30) | ______ |
| Problem 2 (30) | ______ |
| Problem 3 (20) | ______ |
| Problem 4 (20) | ______ |
| Total (100)     | ______ |

Name______________________________
1. Our observation $Y$ is zero-mean Gaussian with variance 1 under $H_0$, and uniformly distributed on (-3,3) under $H_1$.
(a) (10 pts.) Find the maximum likelihood detector, specifying decision regions.
(b) (10 pts.) Assuming $P(H_0) = P(H_1)$ and no penalty is assessed for correct detection, give an example of values for $C_{01}$ (cost of choosing 0 when 1 is true) and $C_{10}$ at which the Bayesian detector first becomes degenerate, that is, it chooses the same hypothesis regardless of the observation $Y$. 
(c) (10 pts.) Design the detector with the best probability of detection possible when the probability of false alarm rate is set to 0.05.
2. A conditionally, jointly Gaussian vector $\mathbf{Y}$ has pdfs which differ in their means. Under hypothesis 0, the mean vector is $[1 \ 1]^T$ and under hypothesis 1, it is $[-1 \ -1]^T$. The covariance matrix in both cases is

$$\Lambda_{\mathbf{Y}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$ 

(a)(10 pts.) Starting from the likelihood ratio, derive the test statistic in $\mathbf{Y}$ in its simplest form.
(b) (10 pts.) If $P(H_0) = 0.6$ and $P(H_1) = 0.4$, find and sketch the decision regions for the minimum probability of error detector.
(c) (10 pts.) Compute the probability of error for the detector of part (b).
3. A Poisson-distributed number of illegal aliens with parameter $\theta$, is crossing the border with Mexico at Tijuana on day $i$. Assume that the numbers crossing on two separate days are independent RVs. These citizens are fleeing political unrest and violence in the USA. The parameter $\theta$ is 50 if the Republicans maintain control of the House of Representatives ($H_0$), and 20 if the Democrats take control ($H_1$).

(a) (10 pts.) Find the Bayesian detector if $P(H_0) = 0.3$, costs for mistakes are equal, no cost is assessed for correct choices, and we observe three days’ illegal border crossing numbers. Put the test in its simplest possible form.
(b) (10 pts.) For large expected values, the Poisson distribution is often approximated by a Gaussian whose mean and variance match the Poisson parameter. Use this approximation to compute $P_M$, the probability of missing hypothesis 1. You may use the property that a sum of independent Poisson variates yields another Poisson.
4. (a) (10 pts.) Show that if the point \((P_F, P_D)\) is on the ROC curve, that point \((1 - P_F, 1 - P_D)\) is also an achievable operating point for a detector under the given conditional distributions.

(b) (10 pts.) The slope of the ROC curve seems always to be positive for our detectors. Demonstrate that this should be true.
Characteristic function: $\phi_X(\omega) = E[e^{j\omega X}]$
Moment generating function: $\theta_X(s) = E[e^{sX}]$
Joint Gaussian RVs:

$$\begin{align*}
p_X(x) &= \frac{1}{(2\pi)^{N/2}|\Lambda_X|^{1/2}} \exp \left\{-1/2(x - \mu_X)^T \Lambda_X^{-1}(x - \mu_X)\right\} \\
\phi_X(\omega) &= \exp \left\{j\omega^T \mu_X - 1/2\omega^T \Lambda_X \omega\right\}
\end{align*}$$

Exponential distribution: $p_X(x) = \theta e^{-\theta x}u(x)$

Poisson distributed counts: $P(X = k) = \frac{e^{-\theta} \theta^k}{k!}$

Binomially-distributed RV:

$$P(X = k) = B(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

Correlation coefficient of $X$ and $Y$: $\rho_{XY} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$

Chernoff bound: $\mathcal{P}(X \geq a) \leq e^{-at} \theta_X(t)$, with $\theta_X(t)$ the moment generating function.

Bienayme-Chebyshov: $\mathcal{P}(X \geq a) \leq \frac{E[\phi(X)]}{\phi(a)}$ for $\phi(x)$ non-negative, symmetric, increasing on $x \geq 0$.

Integration by parts: $\int u \, dv = uv - \int v \, du$

$$\begin{align*}
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} &= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\
&\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} (ad - bc)
\end{align*}$$