EE 60563: Exam 2, Fall 2007
19 November, 2007

Note: Vectors appear in bold-face, and random entities will be capitalized. Notation should be equivalent to what we’ve used in class; please ask about any notational conventions which aren’t clear to you.

Simplify your answers as much as possible for full credit. If it is helpful to have the numerical value for an expression and you don’t have a calculator, you may make a reasonable estimate (for example, $\sqrt{3} \approx 1.73$) and continue to get the simplest answer possible. You will not be penalized, though, for not evaluating expressions in your final answer which require a calculator or tables which have not been supplied (for example, a final answer of the form $1 + e^{-2}$ is OK).

Problem 1 (15) ______
Problem 2 (15) ______
Problem 3 (30) ______
Problem 4 (20) ______
Problem 5 (20) ______
Total (100) ______

Name: ____________________________
1. (15 pts.) Among the formulae on the last sheet of this exam is an expression of
the likelihood ratio test in Bayesian binary hypothesis testing. Starting with this
formula, work “backwards” to find a complete expression of the risk (average cost)
which the Bayesian detector minimizes. Include explanation of any steps other than
simple algebra, and show your progression clearly.
2. (15 pts.) We found in the derivation of the Bayesian LLS estimator that the estimator’s errors are orthogonal to the measurements. Show, by directly evaluating the expected product, that the error of affine \( \hat{X}_{LLS} \) (see formula sheet if necessary) is orthogonal to arbitrary affine transformations of the data \( \mathbf{Y} \). \( \hat{X}_{LLS} \) may be assumed scalar here if you prefer.
3. Each observation $Y_i$ is zero-mean Gaussian with variance 1 under $H_0$, and variance 4 under $H_1$. The \textit{a priori} probabilities of $H_0$ and $H_1$ are $0.6$ and $0.4$, respectively.

(a)(10 pts.) Find the detector from a single $Y_i$ with minimum probability of error, specifying decision regions.
(b) (10 pts.) Compute the probability of error for the detector in part (a).
(c) (10 pts.) If the observation vector is $Y = [Y_0 \ Y_1]^T$, with two conditionally independent observations, each with the conditional pdf above, find the probability of error for the minimum probability of error detector. How should it compare to the answer in (b)?
4. A conditionally jointly Gaussian vector $\mathbf{Y}$ has pdfs which differ in their means. Under hypothesis 0, the mean vector is $[0 \ 0]^T$ and under hypothesis 1, it is $[2 \ 2]^T$. The covariance matrix in both cases is

$$\mathbf{\Lambda}_Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

(a)(10 pts.) Compute and sketch the decision regions for the maximum likelihood detector.
(b)(10 pts.) Compute the probability of detection in a Neyman-Pearson detector if we set $P_f = 0.05$. 
5. A number of systems feature measurements whose information-bearing component \( K_i \) is a Poisson-distributed RV, whose mean \( \theta \) is the information sought. The mean parameter \( \theta \) of the Poisson is exponentially distributed with mean 10.

(a) (10 pts.) From a single observation of \( K_i \), find the Bayesian least-squared error (MMSE) estimate of \( \theta \).
(b) (10 pts.) Using two observations, $K_0$ and $K_1$, which are independent conditioned on $\theta$, find the linear least-squared error (LMMSE) estimate of $\theta$ in its simplest form.
Characteristic function: \( \phi_X(\omega) = E[e^{i\omega X}] \)
Moment generating function: \( \theta_X(s) = E[e^{sX}] \)
Joint Gaussian RVs:
\[
p_X(x) = \frac{1}{(2\pi)^{N/2}|\Lambda_X|^{1/2}} \exp\left\{ -\frac{1}{2}(x - \mu_X)^T \Lambda_X^{-1}(x - \mu_X) \right\}
\]
\[
\phi_X(\omega) = \exp\left\{ j\omega^T \mu_X - \frac{1}{2}\omega^T \Lambda_X \omega \right\}
\]
Gamma function: \( \int_0^\infty x^ke^{-x}dx = \Gamma(k + 1), \Gamma(k + 1) = k! \) for non-negative integers \( k \)
Exponential distribution, with mean \( \frac{1}{\theta} \), variance \( \frac{1}{\theta^2} \): \( p_X(x) = e^{-\theta x}u(x) \)
Poisson distributed counts with mean and variance \( \theta \): \( P(X = k) = \frac{e^{-\theta} \theta^k}{k!} \)
Binomially-distributed RV:
\[
P(X = k) = B(k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k}
\]
Correlation coefficient of \( X \) and \( Y \): \( \rho_{XY} = \frac{E[(X-\mu_X)(Y-\mu_Y)^*]}{\sigma_X \sigma_Y} \)
Chernoff bound: \( P(X \geq a) \leq e^{-at_\theta_X(t)}, \) with \( t_X(t) \) the moment generating function.
Bienayme-Chebyshev: \( P(X \geq a) \leq \frac{E[\varphi(X)]}{\varphi(a)} \) for \( \varphi(x) \) non-negative, symmetric, increasing on \( x \geq 0 \).
Integration by parts: \( \int u dv = uv - \int v du \)
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{(ad - bc)}
\]
Bayesian binary hypothesis testing:
\[
\frac{P(y|H_1)}{P(y|H_0)} \begin{cases} H_1 \iff P_0(C_{10} - C_{00}) \\ H_0 \iff P_1(C_{01} - C_{11}) \end{cases}
\]
(Affine) Linear Least-Squared (LLS) Error Estimation (a.k.a. LMMSE)
\[
\hat{\mathbf{X}}_{LLS}(\mathbf{Y}) = \mathbf{m}_X + \Lambda_{XY} \Lambda_{YY}^{-1}(\mathbf{Y} - \mathbf{m}_Y)
\]