EE 563: Exam 2, Fall 1992

1. Let a sample space be the real numbers, the field of events be the Borel field including the open intervals, and probability measure be such that \( P(\zeta \in (x, x + dx)) = dx \), for \( x \in [0, 1] \). Let the random process \( X(t) \) be defined as \( X(t) = \cos(2\pi|t + \zeta|) \).

(a) (10 pts.) Establish whatever you can concerning the ergodicity (in mean and correlation) of \( X(t) \). In recalling what we discussed in class, be careful of the difference between necessary and sufficient conditions for ergodicity.

(b) (10 pts.) Form the sequence \( X(n) \) by sampling \( X(t) \) at integer times. Show in which senses the sequence \( X(n) \) converges.

2. (35 pts.) The LSI discrete-time system below has impulse response

\[
h(n) = \begin{cases} 
1, & n = -1, 0, 1, \\
0, & \text{elsewhere.} 
\end{cases}
\]

As input, we have the random sequence \( X(n) \) with

\[ E[X(n)] = 2, \forall n, \]

and

\[ R_{XX}(m, n) = 4 + 2\delta(m - n) \]

(a) What are \( E[X^2(2)] \) and \( \sigma^2_{X[2]} \)?

(b) Find \( R_{YY}(m, n) \). Prove what you can concerning stationarity properties of \( Y(n) \).

(c) Write the covariance matrix of the three RVs \( Y(1), Y(2), \) and \( Y(4) \).

(d) What assumptions (if any) are necessary for you to explicitly write the joint probability density function for the RVs \( Y(0) \) and \( Y(1), f_Y(x, y; 0, 1) \)? Make these assumptions and write their PDF.

3. (15 pts.) Use the Bienayme-Chebyshev Inequality to prove the Weak Law of Large Numbers holds for the case of

\[ Y(n) = \frac{1}{n} \sum_{i=1}^{n} X_i, \]

with the \( X_i \) i.i.d. zero-mean, and \( 0 < \sigma^2_x < \infty \). That is, prove that

\[ Y(n) \xrightarrow{P} E[X_i] \]
<table>
<thead>
<tr>
<th>Continuous-time (auto-corr. fn $R_{XX}(t, s)$)</th>
<th>Discrete-time (corr. matrix $R_X$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(n) = \sum_{i=1}^{N} A_i e_i(n)$ for $1 \leq n \leq N$, and orthonormal vector set ${e_i}$</td>
<td>$TT^H = I$, with $T$ being the matrix with eigenvectors of $R_X$ in its rows.</td>
</tr>
<tr>
<td>$\int_a^b R_{XX}(t, s) \psi(s) , ds = \lambda \psi(t)$</td>
<td>[ R_{XX}(t, s) = \sum_{i=1}^{\infty} \lambda_i \psi_i(t) \psi_i^*(s) ]</td>
</tr>
</tbody>
</table>

4. (a) (10 pts.) Write the equivalent expression concerning the Karhunen-Loève transformation for the missing case of continuous or discrete time for the table above.

(b) (10 pts.) Show that any orthonormal set of functions on the interval $(a, b)$ will work for the K-L expansion of a “white,” zero-mean process $X(t)$.

(c) (10 pts.) Suppose now you sample the $X(t)$ of (b) at times $t = 1, 2, 3, 4$ to form the random vector $\mathbf{X}$. Describe the form of the correlation matrix $R_X$ of these RVs. Give an example of a KL expansion for $\mathbf{X}$, and show that it works.