EE 563: Exam 2, Fall 1999
30 November, 1999

The notation on this exam is intended to be identical to that used in lecture. If any of the notation is unclear to you, please ask about it.

Problem 1 (15) _____
Problem 2 (25) _____
Problem 3 (40) _____
Problem 4 (20) _____
Total (100) _____

Name_________________________
1. (15 pts.) $X(t)$ is a strict-sense stationary random process. Prove that its autocorrelation function $R_X(t_1, t_2)$ must reach its maximum at $(0,0)$. Show the corresponding property of the power spectral density.
2. $N(t)$ is a Poisson counting process. According to the standard definition, this means

$$N(t) = \sum_{n=0}^{\infty} U(t - T[n]), \quad t \geq 0,$$

with $T[n]$ being arrivals which form a Poisson point process on $t \geq 0$, and $U(t)$ the unit step function. The resulting RP is discrete-valued, with

$$P_N(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t \geq 0, \quad k = 0, 1, 2, ...$$

(a) (10 pts.) Use the Central Limit Theorem to estimate $P(N(100) \geq 110\lambda)$. 
(b) (15 pts.) Let the RP $Y$ be defined by

$$Y(t) = \frac{N(t)}{\lambda t}.$$  

Prove any sort of convergence you can for the random sequence $Y(n)$, which is just the sequence of samples from $Y(t)$ for all integers $n \geq 0$. (You may use the facts of stronger forms implying weaker forms of convergence.)
3. The RP $Z(t)$ has the following form:

$$Z(t) = A \cos(2\pi t + \theta) + B \sin(4\pi t + \phi) + W(t).$$

The RVs $A, B, \theta, \phi$ are all uniformly distributed on $(-\pi, \pi]$ and mutually independent. $W(t)$ is “white” Gaussian noise, uncorrelated with $A, B, \theta, \phi$ and having $R_W(\tau) = N_0 \delta(\tau)$.

(a) (15 pts.) Find the autocorrelation function of $Z(t)$. 
(b) (10 pts.) Prove directly (without appeal to other theorems) that $Z(t)$ is mean-square ergodic in mean.
(c) (15 pts.) Find a Karhunen-Loève expansion of $Z(t)$ on $(0,2)$, and show that it satisfies the defining integral equation.
4. (20 pts.) The deterministic time function $P(t)$ is shown below. The WSS random process $X(t) = P(t - \Delta)$, where $\Delta$ is uniformly distributed on $(0, T)$. Find the autocorrelation function of $X(t)$ by first establishing characteristics of the power spectral density (allowing generalized Fourier transforms), then considering the implications for the autocorrelation. This does not mean that you must find the exact form of $S_X(\omega)$. (Use basic properties of PSDs and Fourier transform/series properties. You may also use what you’ve seen concerning the PSD/autocorrelation of single sinusoids.)

![Graph of deterministic time function](image-url)
Characteristic function: $\phi_X(\omega) = E[e^{j\omega X}]$

Moment generating function: $\theta_X(s) = E[e^{sX}]$

Joint Gaussian RVs:

$$f_X(x) = \frac{1}{(2\pi)^{N/2}|K_X|^{1/2}} \exp \left\{ -1/2(x - \mu_X)^T K_X^{-1}(x - \mu_X) \right\}$$

$$\phi_X(\omega) = \exp \left\{ j\omega^T \mu_X - 1/2\omega^T K_X \omega \right\}$$

Poisson distributed counts: $P(X = k) = \frac{e^{-\theta_X} \theta_X^k}{k!}$

Correlation coefficient of $X$ and $Y$: $\rho_{XY} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y}$

Chernoff bound: $P(X \geq a) \leq e^{-at} \theta_X(t)$, with $\theta_X(t)$ the moment generating function.

Bienayme-Chebyshev: $P(X \geq a) \leq \frac{E[|X|]}{\varphi(a)}$ for $\varphi(x)$ non-negative, symmetric, increasing on $x \geq 0$.

Mercer’s Theorem: For non-negative definite, Hermitian symmetric $R_X$, for all $t_1, t_2 \in (a, b)$,

$$R_X(t_1, t_2) = \sum_{n=1}^{\infty} \lambda_n \phi_n(t_1) \phi_n(t_2),$$

with functions $\{\phi_n(t)\}$ orthonormal on $(a, b)$ and $\lambda_n \geq 0$, satisfying the Karhunen-Loève Integral Equation:

$$\int_a^b R_X(t_1, t_2) \phi(t_2) dt_2 = \lambda \phi(t_1)$$