EE 60563: Final Exam, Fall 2006
13 December, 2006

Note: Vectors appear in bold-face, and random entities will be capitalized. Notation should be equivalent to what we’ve used in class; please ask about any notational conventions which aren’t clear to you.

If your answer is a density function, be sure to include regions of support in the definition, if they are not the entire space. Simplify your answers as much as possible for full credit. If it is helpful to have the numerical value for an expression and you don’t have a calculator, you may make a reasonable estimate (for example, $\sqrt{3} \approx 1.73$) and continue. You will not be penalized, though, for not evaluating expressions in your final answer which require a calculator or tables which have not been supplied.

You are allowed only a single sheet, 8.5 x 11 inches, with both sides containing whatever notes you wish to bring into the exam.

Please note that this exam has been copied two-sided, so don’t forget to look at both sides of each sheet!

Problem 1 (40) ______

Problem 2 (20) ______

Problem 3 (20) ______

Problem 4 (30) ______

Problem 5 (20) ______

Total (130) ______

Name_________________________________
1. Each of the elements of the vector \( \mathbf{Y} = [Y_0 \ Y_1 \ldots \ Y_{N-1}] \) is conditionally Gaussian, with mean \( M \). The conditional covariance matrix is diagonal, \( \Lambda_{\mathbf{Y}|\mathbf{M}} = \sigma^2 \mathbf{I} \).

   (a) (10 pts.) Find the form of the maximum-likelihood estimate of the vector \( [M \ \sigma^2]^T \) and the Cramér-Rao lower bound on the variance of unbiased estimates of the two parameters.
(b) (10 pts.) Compute the probability $P(\sqrt{Y_0^2 + Y_3^2} \geq 1)$ if $M = 0$. 
(c) (10 pts.) If $N = 5$ and $M = 1$, find $P(\sum_{i=0}^{N} Y_i \leq 8)$
(d) (10 pts.) Suppose $M$ is random, and
\[ p_M(m) = \frac{1}{3}[U(m) - U(m - 3)]. \]

If $N = 1$ and we observe $\{Y_0 = 1\}$, find a posteriori probability density for $M$ in its simplest form, and the linear minimum mean-squared error estimate of $M$ from $Y$. 
2. The conditional distribution of $Y$ takes the form

$$p_Y(y|\theta) = \theta e^{-\theta y}U(y).$$

(a) (10 pts.) Two hypotheses are given as

$$H_0 : \theta = 0.2$$
$$H_1 : \theta = 0.5$$

If $P(H = H_0) = 0.25$ and $P(H = H_1) = 0.75$, there is no penalty for correct decisions, the penalty for choosing $H_0$ when $H = H_1$ is 1, and the penalty for choosing $H_1$ when $H = H_0$ is 2, design the Bayesian detector.
(b) (10 pts.) For the detector you’ve designed in (a), compute its risk, that is its expected cost.
3. (a) (10 pts.) Use the characteristic function to show that the sum of two independent Poisson observations (of means $\theta_1$ and $\theta_2$) is also a Poisson RV.
(b) (10 pts.) If the number of discrete events occurring on any time interval \((t_0, t_0 + T)\) is a Poisson RV with parameter \(\lambda T\), find the probability density function of the random time at which the first event after time \(t=0\) occurs. These events could be, for example, arrivals of customers in a queue.
4. We observe the signal-plus-noise input as follows:

\[ Y_0 = X + W_0 \]
\[ Y_1 = X + W_1 \]

\( X \) is Gaussian(0,1) and \( W \) is Gaussian, independent of \( X \), with mean of zero and the covariance

\[ \Lambda_w = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \]

(a) (10 pts.) Find the Fisher information on \( X \) found in \( Y \).
(b) (10 pts.) Give the simplest form of the least-squared error estimate of $X$ from $Y$. 
(c) (10 pts.) Assuming that \( Y = [Y_0 \ Y_1 \ ...Y_{N-1}]^T \) has the property that each observation, conditioned on \( X \), is independent of all observations except its closest neighbors, and that the covariance is as above in a tri-diagonal matrix, prove that the average of the \( Y_i \)'s is a consistent estimator of \( X \), i.e. that it converges in probability to \( X \).
5. A set of i.i.d. RVs in \( Y \) is exponentially distributed:

\[
p_{Y_i}(y_i) = \frac{1}{\theta} e^{-\frac{y_i}{\theta} U(y_i)}, \quad i = 0, 1, 2, ..., N - 1
\]

(a) (10 pts.) Find the maximum-likelihood estimate of \( \theta \) as a function of the \( N \) observations in \( Y \). What is the variance of the ML estimator? Is it efficient? Support your claim.
(b) (10 pts.) If $\theta$ is $\frac{1}{x}$, and we actually wish to estimate $x$, find the ML estimate of $x$

*directly from your result in (a).* What is the CR bound for unbiased estimates of $\hat{x}$? Does the ML estimate achieve this bound?
Characteristic function: $\phi_X(\omega) = E[e^{j\omega X}]$
Moment generating function: $\theta_X(s) = E[e^{sX}]$
Joint Gaussian RVs:
\[
p_X(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}|\mathbf{\Lambda}_X|^{1/2}} \exp \left\{ -\frac{1}{2}(\mathbf{x} - \mathbf{\mu}_X)^T \mathbf{\Lambda}_X^{-1}(\mathbf{x} - \mathbf{\mu}_X) \right\}
\[
\phi_X(\omega) = \exp \left\{ j\omega^T \mathbf{\mu}_X - \frac{1}{2}\omega^T \mathbf{\Lambda}_X \omega \right\}
\]
Exponential distribution: $p_X(x) = \theta e^{-\theta x} u(x)$, $E[X] = \theta^{-1}$, $Var(X) = \theta^{-2}$.
Poisson distributed counts: $P(X = k) = \frac{e^{-\theta} \theta^k}{k!}$, $E[X] = \theta$, $Var(X) = \theta$.
Binomially-distributed RV:
\[
P(X = k) = B(k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k}
\]
Correlation coefficient of $X$ and $Y$: $\rho_{XY} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y}$
Chernoff bound: $P(X \geq a) \leq e^{-at} \theta_X(t)$, with $\theta_X(t)$ the moment generating function.
Bienayme-Chebyshev: $P(X \geq a) \leq \frac{E[\phi(X)]}{\phi(a)}$ for $\phi(x)$ non-negative, symmetric, increasing on $x \geq 0$.
Integration by parts: $\int u dv = uv - \int v du$
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}
\]