EE 60563: Final Exam, Fall 2015
17 December, 2015

Note: Vectors appear in bold-face, and random entities will be capitalized. Notation should be equivalent to what we’ve used in class. If you feel a problem lacks information necessary to solve what is requested, state carefully any further assumptions you have to make to proceed, then complete the problem.

In any True/False questions, “True” means the statement is true in all cases. Support your response on these questions by providing counterexamples or citing evidence/theorems/axioms that make them true.

Show all but trivial steps in your solutions. Simplify your answers as much as possible for full credit. You will not be penalized, though, for not evaluating expressions which require unsupplied tables or calculators.

This exam is closed-book, closed notes, no calculators, laptops, tablets or phones.

Problem 1 (25) _____
Problem 2 (15) _____
Problem 3 (15) _____
Problem 4 (15) _____
Problem 5 (20) _____
Problem 6 (10) _____
Total (100) _____

Name _______________________________
1. Short stuff (5 pts. each):
   (a) True (T) or False (F): A function that is valid as a characteristic function \( \phi_X(\omega) \) of an RV is also valid as the power spectral density \( S_{XX}(\omega) \) of a WSS random process. Give a brief argument in favor of “T” or a counterexample for “F”.

   (b) The arrival of students at your office hours is a Poisson process with a mean (per hour) of 3 students arriving. What is the expected square of the number of arrivals in an hour?

   (c) Based on the axioms of probability, write a brief proof that, when \( A \subset B \), \( P(A) \leq P(B) \).
(d) Two marginal pdfs are: 
\[ f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2} \right\}, \quad f_Y(y) = \frac{1}{2^{\sqrt{2\pi}}} \exp\left\{ -\frac{y^2}{8} \right\}, \] 
and the correlation coefficient is \( \rho_{XY} = 0 \). True or False: \( X \) and \( Y \) are independent. Explain your answer.

(e) The 2D pdf below is constant-valued within the ellipse and zero elsewhere. Is the correlation coefficient of the two RVs positive or negative? Provide a brief explanation.
2. We have a line of refrigerators whose time to failure is exponentially distributed, with a mean of 5 years, and we have an advertising slogan of “We’re pretty sure his fridge will last for at least 4 years.”
(a) (8 pts) The Consumer Protection Agency checks on us by averaging times from purchase to failure of 100 of our refrigerators. Failure times of individual machines are mutually independent. If the average time to failure falls below 4 years, we will be fined heavily. Use the central limit theorem to estimate the probability that we get fined.

(b) (7 pts) We find one of our fridges that has operated without failure for 9 years. Conditioned on this, what is the probability that it will make it to 10 years?
3. (15 pts.) The system below describes the composition of an RP \( Z(t) \). \( X(t) \) and \( W(t) \) are WSS RPs, both modeled as being “white”, with \( R_{XX}(\tau) = 5\delta(\tau) \) and \( R_{WW}(\tau) = \delta(\tau) \). \( X \) and \( W \) are mutually independent processes. The system \( h(t) \) has impulse response \( e^{-t}U(t) \). Starting from development of auto- and cross-correlations of the processes, find the frequency response of the non-causal Wiener filter for estimating \( X(t) \) from the signal \( Z(t) \). Determine whether the output of the Weiner filter is mean-square differentiable.
4. The jointly Gaussian random vector $\mathbf{X}$ has covariance matrix and mean vector

$$
\mathbf{C}_{\mathbf{XX}} = \begin{bmatrix}
3 & 1 & -1 \\
1 & 3 & 1 \\
-1 & 1 & 2
\end{bmatrix}, \mathbf{m}_X = \begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix}
$$

(a) (8 pts) We have the RV $Y = 2X_1 + X_3$. Write explicitly the pdf $f_Y(y)$ in its simplest form.

(b) (7 pts.) Find the minimum mean-squared error estimator of $X_1$ from observation of $Y$.  

5. The solution below comes from HW 9, concerning the orthogonality of basis vectors in the K-L expansion of an RP. Gubner’s solution is based on the assumption that the RP and all the basis vectors are strictly real-valued. This may not always be the case; Fourier series components may be an option, for example.

31. To begin, write

\[ \lambda_k \int_a^b \varphi_k(t) \varphi_m(t) \, dt = \int_a^b \lambda_k \varphi_k(t) \varphi_m(t) \, dt = \int_a^b \left[ \int_a^b R(t,s) \varphi_k(s) \, ds \right] \varphi_m(t) \, dt \]

\[ = \int_a^b \varphi_k(s) \left[ \int_a^b R(s,t) \varphi_m(t) \, dt \right] \, ds, \quad \text{since } R(t,s) = R(s,t), \]

\[ = \int_a^b \varphi_k(s) \cdot \lambda_m \varphi_m(s) \, ds = \lambda_m \int_a^b \varphi_k(s) \varphi_m(s) \, ds. \]

We can now write

\[ (\lambda_k - \lambda_m) \int_a^b \varphi_k(i) \varphi_m(t) \, dt = 0. \]

If \( \lambda_k \neq \lambda_m \), we must have \( \int_a^b \varphi_k(t) \varphi_m(t) \, dt = 0. \)

(a) (10 pts) Generalize the result above by applying a “*” for conjugation wherever it would belong for the general case. (Mark this directly on the equations above.) When you’re finished, you should have the option for a stronger result, extracting two key characteristics of K-L decompositions from this development. State clearly the two results.
31. To begin, write

$$\lambda_k \int_a^b \varphi_k(t) \varphi_m(t) \, dt = \int_a^b \lambda_k \varphi_k(t) \varphi_m(t) \, dt = \int_a^b \left[ \int_a^b R(t,s) \varphi_k(s) \, ds \right] \varphi_m(t) \, dt$$

$$= \int_a^b \varphi_k(s) \left[ \int_a^b R(s,t) \varphi_m(t) \, dt \right] \, ds, \quad \text{since } R(t,s) = R(s,t),$$

$$= \int_a^b \varphi_k(s) \cdot \lambda_m \varphi_m(s) \, ds = \lambda_m \int_a^b \varphi_k(s) \varphi_m(s) \, ds.$$

We can now write

$$(\lambda_k - \lambda_m) \int_a^b \varphi_k(t) \varphi_m(t) \, dt = 0.$$

If $\lambda_k \neq \lambda_m$, we must have $\int_a^b \varphi_k(t) \varphi_m(t) \, dt = 0$.

(b)(10 pts.) Repeat the development of (a) for the finite-dimensional case, in which we analyze the eigendecomposition of a correlation matrix. Replace infinite-dimensional operations above with their finite-dimensional counterparts featuring eigenvectors $\{e_i\}$, etc. Be sure you get all your transpositions and conjugations in the right places, and state the results for this case similarly to those in (a).
6. A probability space is defined on the real line indexed by \( \omega \), with the Borel field containing all open intervals as the events to be measured, and probability measure specified as

\[
P(\omega \in (a, b)) = \begin{cases} 
  b - a & 0 < a \leq b < 1 \\
  0 & a, b \notin (0, 1) 
\end{cases}.
\]

(a) (5 pts.) A discrete-time RP is defined as

\[
X(n; \omega) = \begin{cases} 
  \sqrt{n} \cos(2\pi \omega n) & 0 < |\omega| < 1/n \\
  0 & \text{elsewhere.}
\end{cases}
\]

In what senses does this sequence converge, and to what RV? Show your computation/argument for your results for all the senses we’ve discussed.
(b) (5 pts.) If $\theta$ is uniformly distributed on $[0, 2\pi]$, independent of $\omega$ and we define the DT RP $Y[n] = \sin(2\pi\omega n + \theta)$, establish whether $Y[n]$ is wide-sense stationary.