ON SOME INVARIANTS OF ORBITS IN THE FLAG VARIETY UNDER A SYMMETRIC SUBGROUP

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Abstract. Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic not equal to 2, let $\mathcal{B}$ be the variety of all Borel subgroups of G, and let K be a symmetric subgroup of G. Fixing a closed K-orbit in $\mathcal{B}$, we associate to every K-orbit on $\mathcal{B}$ some subsets of the Weyl group of G, and we study them as invariants of the K-orbits. When $\mathbf{k} = \mathbb{C}$, these invariants are used to determine when an orbit of a real form of G and an orbit of a Borel subgroup of G have non-empty intersection in $\mathcal{B}$. We also characterize the invariants in terms of admissible paths in the set of K-orbits in $\mathcal{B}$.

1. Introduction

Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic not equal to 2, and let $\mathcal{B}$ be the variety of Borel subgroups of G with the conjugation action by G. Let θ be an order 2 automorphism of G. The θ-fixed point subgroup $K = G^\theta$ acts on $\mathcal{B}$ with finitely many orbits [12, 13, 16].

Let V be the finite set of K-orbits in $\mathcal{B}$, and for $v \in V$, let $K(v) \subset \mathcal{B}$ be the corresponding K-orbit in $\mathcal{B}$. For a subset X of $\mathcal{B}$, let $\overline{X}$ be the Zariski closure of X in $\mathcal{B}$. The Bruhat order on V, denoted by $\leq$, is defined by

$$v_1 \leq v_2 \quad \text{if} \quad K(v_1) \subset \overline{K(v_2)}, \qquad v_1, v_2 \in V. \tag{1.1}$$

When $\mathbf{k} = \mathbb{C}$, the geometry of the K-orbits and their closures in $\mathcal{B}$ plays an important role in the representation theory of real forms of G via the Beilinson-Bernstein correspondence (see [9]). The poset $(V, \leq)$ and its application to representation theory have been the focus of extensive studies (see [1, 2, 3, 12, 13, 16, 17, 20]).

Let W be the canonical Weyl group of G with the set S of canonical generators, and let $M(W, S)$ be the corresponding monoid (see §3.3 and §3.6). Among the structures on V are the *monoidal action* of $M(W, S)$ on V (see §3.6), the *cross action* of W on V (see §6.5), and the map $\phi : V \to W$ defined by Springer (see §6.6). The programs from the *Atlas of Lie groups* (www.liegroups.org) facilitate explicit computations in examples of the cross action, the monoidal action, and the Springer map.

Let $V_0 = \{v \in V : K(v) \subset \mathcal{B} \text{ is closed}\}$. In this paper, we associate to every $v_0 \in V_0$ and $v \in V$ three subsets $Z_{v_0}(v) \subset Y_{v_0}(v) \subset W_{v_0}(v)$ of the Weyl group W. The set $W_{v_0}(v)$ is defined both geometrically in terms of intersections of $K(v)$ and Schubert varieties in $\mathcal{B}$ and combinatorially in terms of the Bruhat order on V and the monoidal action of $M(W, S)$ on V, and $Y_{v_0}(v)$ (resp. $Z_{v_0}(v)$) is defined to be the set of minimal (resp. minimal length) elements in $W_{v_0}(v)$.

One of our main results (see Theorem 2.2 and Corollary 2.3) says that for any $v_0 \in V_0$, the subset $Y_{v_0}(v)$ (or $Z_{v_0}(v)$) of W, together with the Springer invariant $\phi(v) \in W$, completely determine $v \in V$. This result generalizes [13, Theorem 5.2.2] of Richardson-Springer in the case when (G, θ) is of Hermitian symmetric type (see §9.3). When V_0 contains a single element v_0, the set $Y_{v_0}(v)$ for any $v \in V$ is a special case of an invariant for $K(v)$ studied by Springer in [17].

When $k = \mathbb{C}$ and K is the complexification of a maximal compact subgroup of a real form G_0 of G, we use the Matsuki duality between K-orbits and G_0-orbits in $\mathcal{B}$ to determine when a G_0-orbit and a Bruhat cell in $\mathcal{B}$ have non-empty intersection. The problem of determining when two such orbits intersect comes from Poisson geometry and served as the main motivation for this paper. See §2.3 and §5.

We introduce the notion of *admissible paths* from $v_0 \in V_0$ to $v \in V$ by allowing only certain types of cross and monoidal actions by simple generators of W. We then characterize elements in $Y_{v_0}(v)$ (resp. $Z_{v_0}(v)$) as products of the simple generators in *minimal* (resp. *shortest*) admissible paths from v_0 to v. See §2.5 and §8.

The technical part of the paper is an analysis in §7 on the sets $Y_{v_0}(v)$ in relation to simple roots and their types relative to v. Various examples are computed in §9.

Precise statements of the main results in the paper are given in §2.

Most of the structures on V are defined in [12, 13, 16] using a *standard pair*, i.e., a pair (B, H), where $B \in \mathcal{B}$ and $H \subset B$ a maximal torus of G such that $\theta(B) = B$ and $\theta(H) = H$. However, to formulate the sets $Z_{v_0}(v) \subset Y_{v_0}(v) \subset W_{v_0}(v)$, it is crucial that these structures can be defined independently of the standard pairs. Following [13, §1.7], we review in §3 and §6 the canonical definitions of all the structures on V needed in the paper.

Acknowledgments. We would like to thank Xuhua He and George McNinch for helpful discussions. We thank the organizers of *the Atlas of Lie Groups* meeting in Utah in July 2010, which helped the development of this paper, and also thank Jeff Adams, Peter Trapa, and David Vogan for very useful discussions. We are especially grateful to Scott Crofts, who wrote a program allowing us to compute the invariants $Z_{v_0}(v)$ and $Y_{v_0}(v)$ explicitly in examples. The work described in this paper by the first author was partially supported by NSA grants H98230-08-0023 and H98230-11-1-0151, and that by the second author by the RGC grants HKU 7034/05P and 7037/07P of the Hong Kong SAR, China.

2. Statements of Results

2.1. **Geometric and combinatorial definitions of $W_{v_0}(v) \subset W$.** For $v_0 \in V_0$ and $v \in V$, we define $W_{v_0}(v) \subset W$ by

$$W_{v_0}(v) = \{w \in W : K(v) \cap \overline{B(w)} \neq \emptyset\}, \tag{2.1}$$

where B is any Borel subgroup of G contained in $K(v_0)$, and for $w \in W$, $B(w)$ is the corresponding B-orbit in $\mathcal{B}$ (see §3.5). The set $W_{v_0}(v)$ depends only on v_0 and v and not on the choice of $B \in K(v_0)$ (see Lemma 4.1).

Denote the monoidal action of $M(W, S) = \{m(w) : w \in W\}$ on V by $m(w)\cdot v$ for $w \in W$ and $v \in V$ (see §3.6). Our Lemma 4.2 gives the following combinatorial description of the set $W_{v_0}(v)$ for $v_0 \in V$ and $v \in V$:

$$W_{v_0}(v) = \{w \in W : \ v \leq m(w)\cdot v_0\} \subset W, \tag{2.2}$$

where recall that $\leq$ is the Bruhat order on V defined in (1.1).

2.2. **The subsets $Z_{v_0}(v) \subset Y_{v_0}(v)$ of $W_{v_0}(v)$.** Let $l : W \to \mathbb{N}$ and $\leq$ be respectively the length function and the Bruhat order on W as a Coxeter group (see §3.3 and §3.4). Let W_1 be a subset of W. An element $w \in W_1$ is said to be *minimal* if for any $w_1 \in W_1$, $w_1 \leq w$ implies that $w_1 = w$. The set of all minimal elements in W_1 will be denoted by $\min(W_1)$. An element $w \in W_1$ is said to have *minimal length* if $l(w) \leq l(w_1)$ for every $w_1 \in W_1$. The set of all minimal length elements in W_1 is denoted by $\min_l(W_1)$.

For $v_0 \in V_0$ and $v \in V$, define

$$Y_{v_0}(v) = \min(W_{v_0}(v)) \subset W_{v_0}(v), \tag{2.3}$$

$$Z_{v_0}(v) = \min_l(W_{v_0}(v)) = \min_l(Y_{v_0}(v)) \subset Y_{v_0}(v). \tag{2.4}$$

We prove in Lemma 4.3 that for any $v_0 \in V$ and $v \in V$, $W_{v_0}(v)$ is determined by its subset $Y_{v_0}(v)$ in the sense that for any $w \in W$,

$$w \in W_{v_0}(v) \quad \text{iff} \quad y \leq w \ \text{ for some } y \in Y_{v_0}(v). \tag{2.5}$$

Thus both $Z_{v_0}(v)$ and $W_{v_0}(v)$ are determined by $Y_{v_0}(v)$. The example in §9.4 shows that $Z_{v_0}(v)$ may be a proper subset of $Y_{v_0}(v)$.

Using the programs available at the *Atlas of Lie groups*, Scott Crofts has written a program that allows one to compute the sets $Y_{v_0}(v)$ explicitly in examples.

2.3. **Intersections of real group orbits and Bruhat cells in $\mathcal{B}$.** Assume now that $\mathbf{k} = \mathbb{C}$ and that K is the complexification of a maximal compact subgroup of a real form G_0 of G. For $v \in V$, let $G_0(v) \subset \mathcal{B}$ be the G_0-orbit in $\mathcal{B}$ that is dual to $K(v) \subset \mathcal{B}$ under the Matsuki duality [10] between G_0-orbits and K-orbits in $\mathcal{B}$. The following Theorem 2.1 is the first main result of this paper.

Theorem 2.1. *Let $v_0 \in V_0$ and let $B \in K(v_0)$. Then for $v \in V$ and $w \in W$,*

$$G_0(v) \cap B(w) \neq \emptyset \quad \textit{iff} \quad w \in W_{v_0}(v).$$

Our motivation for Theorem 2.1 comes from Poisson geometry: when G_0 is connected, it is shown in [7] that there is a Poisson structure Π on $\mathcal{B}$ such that the connected components of intersections of G_0-orbits and B-orbits in $\mathcal{B}$ are precisely the H_0-orbits of symplectic leaves of Π, where $H_0 = B \cap G_0$ is a maximally compact Cartan subgroup of G_0. Thus, one first needs to know when the intersection of a G_0-orbit and a B-orbit in $\mathcal{B}$ is non-empty. In view of (2.5), Theorem 2.1 gives a complete answer to this question in terms of the set $Y_{v_0}(v)$ and the Bruhat order on W. Further applications to the Poisson structure Π on $\mathcal{B}$ will appear elsewhere.

2.4. $Y_{v_0}(v)$ **and** $Z_{v_0}(v)$ **as invariants of** $K(v)$**.** Our second main result, the following Theorem 2.2, implies that the pair $(\phi(v), Y_{v_0}(v))$ forms a complete invariant for $v \in V$, where $\phi : V \to W$ is the Springer map (see §6.6).

Theorem 2.2. *Let $v_0 \in V_0$ and $v, v' \in V$. If $\phi(v) = \phi(v')$ and $Y_{v_0}(v) \cap Y_{v_0}(v') \neq \emptyset$, then $v = v'$.*

Since $Z_{v_0}(v) \subset Y_{v_0}(v)$ for any $v_0 \in V_0$ and $v \in V$, one also has

Corollary 2.3. *Let $v_0 \in V_0$ and $v, v' \in V$. If $\phi(v) = \phi(v')$ and $Z_{v_0}(v) \cap Z_{v_0}(v') \neq \emptyset$, then $v = v'$.*

2.5. **Subexpressions and admissible paths.** Reduced decompositions for elements in V and their subexpressions are introduced in [12, §5] and [13, §3, §4] (and see §6.11 for details). In particular, if $v, v' \in V$ and if $(\mathbf{v}', \mathbf{s}')$ is any reduced decomposition of v', then [13, Proposition 4.4] $v \leq v'$ if and only if there exists a subexpression of $(\mathbf{v}', \mathbf{s}')$ with final term v.

Let $v_0 \in V_0$ and $v \in V$. We show that if $y \in Y_{v_0}(v)$, then every reduced decomposition of $m(y) \cdot v_0$ coming from a reduced word of y has *exactly one* subexpression with final term v. See Proposition 7.6 for details.

For $v_0 \in V_0$ and $v \in V$, we define an *admissible path* from v_0 to v to be a pair $(\mathbf{v}, \mathbf{s})$, where $\mathbf{v} = (v_0, v_1, \ldots, v_k)$ is a sequence in V and $\mathbf{s} = (s_1, \ldots, s_k)$ is a sequence in S, such that for each $j \in [1, k]$, v_j is a certain type of either the cross action or the monoidal action of s_j on v_{j-1} (see Definition 8.1). For such a path $(\mathbf{v}, \mathbf{s})$, let $y(\mathbf{v}, \mathbf{s}) = s_k \cdots s_1 \in W$. We also define the set $\mathcal{P}_{\min}(v_0, v)$ (resp. $\mathcal{P}_{\mathrm{short}}(v_0, v)$) of *minimal* (resp. *shortest*) paths from v_0 to v. We prove (see Corollary 8.8 and Corollary 8.10) that for any $v_0 \in V_0$ and $v \in V$,

$$Y_{v_0}(v) = \{y(\mathbf{v}, \mathbf{s}) : (\mathbf{v}, \mathbf{s}) \in \mathcal{P}_{\min}(v_0, v)\}$$
$$Z_{v_0}(v) = \{y(\mathbf{v}, \mathbf{s}) : (\mathbf{v}, \mathbf{s}) \in \mathcal{P}_{\mathrm{short}}(v_0, v)\}.$$

We believe that the minimal and shortest paths defined in this paper will have other applications to the study of K-orbit closures in $\mathcal{B}$.

3. Review on K-orbits in $\mathcal{B}$, I

Following [13, §1.7], we review in this section the canonical Weyl group W of G with its set S of canonical generators and the action of the monoid $M(W, S)$ on V. More structures on V will be reviewed in §6.

3.1. **Notation.** If Q is a subgroup of G and $g \in G$, Q^g denotes the subgroup $g^{-1}Qg$ of G. We will consider the right action of G on the flag variety $\mathcal{B}$ by $\mathcal{B} \times G \to \mathcal{B}$: $(B, g) \mapsto B^g$ for $B \in \mathcal{B}$ and $g \in G$.

If a group L acts on a set X from the left (resp. right), we will denote by $L\backslash X$ (resp. X/L) the set of L-orbits on X.

Throughout the paper and unless otherwise specified, for a subset X of G (resp. $\mathcal{B}$ and $\mathcal{B} \times \mathcal{B}$), $\overline{X}$ denotes the Zariski closure of X in G (resp. $\mathcal{B}$ and $\mathcal{B} \times \mathcal{B}$).

3.2. **The variety $\mathcal{C}$.** Let $\mathcal{C}$ be the set of all pairs (B, H), where $B \in \mathcal{B}$ and $H \subset B$ is a maximal torus of G. Then G acts on $\mathcal{C}$ transitively from the right by

$$\mathcal{C} \times G \longrightarrow \mathcal{C}: \quad (B, H)^g = (B^g,\ H^g), \qquad (B, H) \in \mathcal{C},\ g \in G. \tag{3.1}$$

The stabilizer subgroup of G at $(B, H) \in \mathcal{C}$ is H. Thus for each $(B, H) \in \mathcal{C}$, one has the G-equivariant identification

$$C_{B,H}: \quad H\backslash G \longrightarrow \mathcal{C}: \quad Hg \longmapsto (B^g, H^g), \qquad g \in G. \tag{3.2}$$

For $(B, H) \in \mathcal{C}$, let $N_G(H)$ be the normalizer of H in G, let $W_H = N_G(H)/H$ be the Weyl group of (G, H), and let $S_{B,H}$ be the set of generators of W_H defined by the simple roots of B relative to H.

Let $(B, H), (B', H') \in \mathcal{C}$. Let $g \in G$ be such that $B' = B^g$ and $H' = H^g$. Since g is unique up to the left multiplication by an element in H, one has a well-defined isomorphism of tori

$$T_{B,H}^{B',H'}: \quad H \longrightarrow H': \quad h \longmapsto g^{-1}hg, \qquad h \in H. \tag{3.3}$$

Moreover, although the map $N_G(H) \to N_G(H') : n \to g^{-1}ng$ depends on the choice of g, the group isomorphism

$$\eta_{B,H}^{B',H'}: \quad W_H \longrightarrow W_{H'}: \quad nH \longmapsto g^{-1}ngH', \qquad n \in N_G(H)$$

does not and is thus well-defined. Since $\eta_{B,H}^{B',H'}(S_{B,H}) = S_{B',H'}$, one has the isomorphism $\eta_{B,H}^{B',H'} : (W_H, S_{B,H}) \to (W_{H'}, S_{B',H'})$ of Coxeter groups.

3.3. **The canonical Weyl group.** Let $W = (\mathcal{B} \times \mathcal{B})/G$ be the set of G-orbits on $\mathcal{B} \times \mathcal{B}$ for the diagonal G-action, and let $p : \mathcal{B} \times \mathcal{B} \to W$ be the natural projection. Let $(B, H) \in \mathcal{C}$. Then the map

$$\eta_{B,H}: \quad W_H \longrightarrow W: \quad nH \longmapsto p(B^n, B), \qquad n \in N_G(H) \tag{3.4}$$

is bijective. It is straightforward to check that for another $(B', H') \in \mathcal{C}$,

$$\eta_{B',H'}^{-1} \circ \eta_{B,H} = \eta_{B,H}^{B',H'} : \quad W_H \longrightarrow W_{H'}.$$

Thus there is a well-defined group structure on W such that $\eta_{B,H} : W_H \to W$ is a group isomorphism for every $(B, H) \in \mathcal{C}$. Let $S = \eta_{B,H}(S_{B,H}) \subset W$ for any $(B, H) \in \mathcal{C}$. Then S is a set of generators for W and is independent of the choice of $(B, H) \in \mathcal{C}$. The Coxeter group (W, S) is called the *canonical Weyl group* of G.

For $w \in W$, a reduced word of w is a shortest expression $w = s_1 \cdots s_{l(w)}$ of w as a product of elements in S, and $l(w)$ is called the length of w.

3.4. **The Bruhat order on W.** For $w \in W$, let

$$\mathcal{O}(w) = \{(B', B) \in \mathcal{B} \times \mathcal{B} : \, p(B', B) = w\} \subset \mathcal{B} \times \mathcal{B}$$

be the corresponding G-orbit in $\mathcal{B} \times \mathcal{B}$. The Bruhat order on W is defined by

$$w \leq w' \quad \text{if} \quad \mathcal{O}(w) \subset \overline{\mathcal{O}(w')}, \quad w, w' \in W.$$

If $w \leq w'$ and $w \neq w'$, we will write $w < w'$ or $w' > w$. It is well-known [6, Théorème 3.13] that for $w, w' \in W$, $w' \leq w$ if and only if there is a reduced word $w = s_1 \cdots s_{l(w)}$ of w such that $w' = s_{i_1} \cdots s_{i_p}$ for some $1 \leq i_1 < \cdots < i_p \leq l(w)$.

For a subset W_1 of W, recall from §2.2 that $\min(W_1)$ is the set of minimal elements in W_1 with respect to the Bruhat order and that $\min_l(W_1)$ is the set of minimal length elements in W_1. The following Lemma 3.1 will be used in the proof of Proposition 4.5.

Lemma 3.1. *Let W_1 and W_2 be two subsets of W such that $\min(W_2) \subset W_1 \subset W_2$. Then $\min(W_1) = \min(W_2)$ and $\min_l(W_1) = \min_l(W_2)$.*

3.5. **Orbits in $\mathcal{B}$ under a Borel subgroup.** Let $B \in \mathcal{B}$. The set of B-orbits in $\mathcal{B}$ is naturally indexed by W. Indeed, for $w \in W$, let

$$B(w) = \{B' \in \mathcal{B} : \, p(B', B) = w\} \subset \mathcal{B}. \tag{3.5}$$

Then $B(w)$ is a single B-orbit in $\mathcal{B}$, and the map $w \mapsto B(w)$ is a bijection from W to the set of all B-orbits in $\mathcal{B}$.

Define $q_B : G \to \mathcal{B}$ by $q_B(g) = B^g$ for $g \in G$. For $w \in W$, let

$$BwB = q_B^{-1}(B(w)) = \{g \in G : p(B^g, B) = w\} \subset G. \tag{3.6}$$

Then BwB be the single (B, B)-double coset in G. Moreover, for $w, w' \in W$,

$$w \leq w' \quad \text{iff} \quad B(w) \subset \overline{B(w')} \quad \text{iff} \quad BwB \subset \overline{Bw'B}.$$

3.6. **The monoidal action of $M(W,S)$ on V.** Our references for this subsection are [12, §4] and [13, §3]. The monoid $M(W,S)$ associated to the Coxeter group (W,S) is $M(W,S) = \{m(w) : w \in W\}$ as a set, with the monoidal product given by

$$m(s)m(w) = \begin{cases} m(sw), & \text{if } sw > w \\ m(w), & \text{if } sw < w \end{cases}, \qquad s \in S,\ w \in W. \tag{3.7}$$

Alternatively, let $B \in \mathcal{B}$, and let $BwB = q_B^{-1}(B(w)) \subset G$ for $w \in W$ as in (3.6). Define $* : W \times W \to W$ such that

$$\overline{B(w_1 * w_2)B} = \overline{(Bw_1B)(Bw_2B)}, \qquad w_1, w_2 \in W. \tag{3.8}$$

Then $*$ is a monoidal product on W, independent of the choice of $B \in \mathcal{B}$, and $(W,*) \to M(W,S) : w \mapsto m(w)$ is an isomorphism of monoids [6, Proposition 3.18].

Let $q_K : G \to G/K$ be the natural projection. Let $B \in \mathcal{B}$, and for $v \in V$, let

$$BvK = q_B^{-1}(K(v)) \subset G.$$

Then $q_K(BvK)$, being a B-orbit in G/K, is irreducible for each $v \in V$. For $w \in W$ and $v \in V$, since $q_K((BwB)(BvK)) \subset G/K$ is irreducible and is a finite union of B-orbits in G/K, there is a unique element in V, denoted by $m(w){\cdot}v$, such that $\overline{q_K((BwB)(BvK)} = \overline{q_K(B(m(w){\cdot}v)K)}$, where $^-$ denotes the Zariski closure in G/K. Consequently, one has, for any $w \in W$ and $v \in V$,

$$\overline{B(m(w){\cdot}v)K} = \overline{(BwB)(BvK)}. \tag{3.9}$$

Lemma 3.2. [12, §4] *The map*

$$M(W,S) \times V \longrightarrow V : \quad (m(w), v) \longmapsto m(w){\cdot}v, \qquad w \in W, v \in V, \tag{3.10}$$

is independent of the choice of $B \in \mathcal{B}$ and defines a monoidal action of $M(W,S)$ on V.

Proof. Let $B \in \mathcal{B}, g_0 \in G$, and $B' = B^{g_0}$. It follows from definitions that for any $w \in W$ and $v \in V$, $B'wB' = g_0^{-1}BwBg_0$ and $B'vK = g_0^{-1}BvK$. Thus, the map in (3.10) is independent of the choice of B. Choosing any $B \in \mathcal{B}$ and using the isomorphism $(W,*) \to M(W,S)$ of monoids, one sees from (3.8) and (3.9) that the map in (3.10) defines an action of $M(W,S)$ on V.

Q.E.D.

Lemma 3.3. *For any $B \in \mathcal{B}, v \in V$, and $w \in W$, one has*

$$\overline{B(m(w){\cdot}v)K} = \overline{BwB}\ \overline{BvK}.$$

Proof. Let $w = s_1 s_2 \cdots s_l$ be a reduced word, and for each $1 \le j \le l$, let $P_j = B \cup Bs_jB$ so that P_j is a parabolic subgroup of G. Then $\overline{BwB} = P_1P_2 \cdots P_j$ (see, for example [6, Théorème 3.13]). By repeatedly applying [6, Lemma 2.12], one sees that $\overline{BwB}\ \overline{BvK}$ is closed in G. Thus $\overline{BwB}\ \overline{BvK} = \overline{(BwB)(BvK)} = \overline{B(m(w){\cdot}v)K}$.

Q.E.D.

Recall that $V_0 = \{v_0 \in V : K(v_0) \subset \mathcal{B} \text{ is closed}\}$. For a subset X of $\mathcal{B}$, let

$$X \cdot K = \{B^k : \ B \in X, k \in K\}.$$

Lemma 3.4. *Let $v_0 \in V_0$ and let $B \in K(v_0)$. Then for any $w \in W$,*

$$\overline{K(m(w) \cdot v_0)} = \overline{B(w) \cdot K} = \overline{B(w)} \cdot K.$$

Proof. Since $K(v_0)$ is closed in $\mathcal{B}$, the double coset $Bv_0K = BK = q_B^{-1}(K(v_0))$ is closed in G. By Lemma 3.3, one has

$$\overline{B(m(w) \cdot v_0)K} = \overline{BwB}\ \overline{BK} = \overline{BwB}\ K.$$

Applying $q_B : G \to \mathcal{B}$, one proves Lemma 3.4.

Q.E.D.

Lemma 3.5. *If $w, w' \in W$ and $v, v' \in V$ are such that $w \leq w'$, $v \leq v'$, then*

$$m(w) \cdot v \leq m(w') \cdot v \quad \text{and} \quad m(w) \cdot v \leq m(w) \cdot v'.$$

Proof. This is immediate from Lemma 3.3.

Q.E.D.

4. The geometrical and combinatorial definitions of $W_{v_0}(v)$

4.1. **The two definitions of $W_{v_0}(v)$.** Recall from §2.1 that for $v_0 \in V_0$ and $v \in V$, the subset $W_{v_0}(v)$ of W is defined by

$$W_{v_0}(v) = \{w \in W : K(v) \cap \overline{B(w)} \neq \emptyset\},$$

where B is any Borel subgroup of G contained in $K(v_0)$.

Lemma 4.1. *For any $v_0 \in V$ and $v \in V$, the set $W_{v_0}(v)$ is independent of the choice of $B \in K(v_0)$.*

Proof. Let $B, B' \in K(v_0)$ and let $B' = B^k$ for some $k \in K$. Let $w \in W$. Then $B'(w) = \{B_1^k : B_1 \in B(w)\}$. Thus $K(v) \cap \overline{B'(w)} \neq \emptyset$ if and only if $K(v) \cap \overline{B(w)} \neq \emptyset$.

Q.E.D.

We now have the following combinatorial interpretation of $W_{v_0}(v)$.

Lemma 4.2. *For any $v_0 \in V_0$ and $v \in V$, one has*

$$W_{v_0}(v) = \{w \in W : \ v \leq m(w) \cdot v_0\}.$$

Proof. Let $w \in W$. By Lemma 3.4 and by the definition of the Bruhat order on V, $v \leq m(w) \cdot v_0$ if and only if $K(v) \subset \overline{K(m(w) \cdot v_0)} = \overline{B(w)} \cdot K$, which is equivalent to $K(v) \cap \overline{B(w)} \neq \emptyset$.

Q.E.D.

4.2. **Properties of $W_{v_0}(v)$.** Recall from §2.2 that for $v_0 \in V_0$ and $v \in V$, $Y_{v_0}(v)$ is the set of minimal elements in $W_{v_0}(v)$ with respect to the Bruhat order on W.

Lemma 4.3. *Let $v_0 \in V_0$, $v \in V$, and $w \in W$. Then $w \in W_{v_0}(v)$ if and only if $w \geq y$ for some $y \in Y_{v_0}(v)$.*

Proof. It follows from the definition of $Y_{v_0}(v)$ that $w \geq y$ for some $y \in Y_{v_0}(v)$. Conversely, assume that $w \geq y$ for some $y \in Y_{v_0}(v)$. Then $K(v) \cap \overline{B(y)} \neq \emptyset$ and $\overline{B(y)} \subset \overline{B(w)}$. Thus $K(v) \cap \overline{B(w)} \neq \emptyset$.

Q.E.D.

Lemma 4.4. *For $v_0 \in V_0$ and $v \in V$, one has $W_{v_0}(v) = W$ if and only if $v = v_0$.*

Proof. Let 1 be the identity element of W. It is clear that $1 \in W_{v_0}(v_0)$, so $W_{v_0}(v_0) = W$ by Lemma 4.3. Assume that $v \in V$ is such that $W_{v_0}(v) = W$. Then $1 \in W_{v_0}(v)$, so $v \leq v_0$. Since $K(v_0) \subset \mathcal{B}$ is closed, we have $v = v_0$.

Q.E.D.

4.3. **The set $W'_{v_0}(v)$.** Fix $v_0 \in V_0$ and let $B \in K(v_0)$. For $v \in V$, let

$$W'_{v_0}(v) = \{w \in W : \; K(v) \cap B(w) \neq \emptyset\}. \tag{4.1}$$

By the proof of Lemma 4.1, $W'_{v_0}(v)$ is independent of the choice of B in $K(v_0)$. One also sees from the definition of $B(w)$ in (3.5) that for any $v_0 \in V_0$ and $v \in V$,

$$W'_{v_0}(v) = \{w \in W : \; w = p(B', B) \text{ for some } B' \in K(v), \; B \in K(v_0)\}. \tag{4.2}$$

The following Proposition 4.5 expresses $Y_{v_0}(v)$ and $Z_{v_0}(v)$ in terms of $W'_{v_0}(v)$.

Proposition 4.5. *For any $v_0 \in V_0$ and $v \in V$, one has*

$$Y_{v_0}(v) \subset W'_{v_0}(v) \subset W_{v_0}(v). \tag{4.3}$$

Consequently, $Y_{v_0}(v) = \min(W'_{v_0}(v))$ and $Z_{v_0}(v) = \min_l(W'_{v_0}(v))$.

Proof. Clearly $W'_{v_0}(v) \subset W_{v_0}(v)$. Let $w \in Y_{v_0}(v)$. Then $K(v) \cap \overline{B(w)} \neq \emptyset$ and $K(v) \cap B(w') = \emptyset$ for any $w' \in W$ such that $w' < w$. Thus $K(v) \cap B(w) \neq \emptyset$ and $w \in W'_{v_0}(v)$. By Lemma 3.1, $Y_{v_0}(v) = \min(W'_{v_0}(v))$, and $Z_{v_0}(v) = \min_l(W'_{v_0}(v))$.

Q.E.D.

Example 4.6. Let $\tilde{G} = G \times G$ and let

$$\tilde{\theta} : \;\; \tilde{G} \longrightarrow \tilde{G} : \quad \tilde{\theta}(g_1, g_2) = (g_2, g_1), \qquad (g_1, g_2) \in \tilde{G}.$$

Then the fixed point subgroup $\tilde{K}$ of $\tilde{\theta}$ in $\tilde{G}$ is $\tilde{K} = \{(g, g) : g \in G\}$, so the set $\tilde{V}$ of $\tilde{K}$-orbits in $\tilde{\mathcal{B}} = \mathcal{B} \times \mathcal{B}$ is W. For $w \in W$, let $\tilde{K}(w) = \mathcal{O}(w)$, where we recall that $\mathcal{O}(w)$ is the G-orbit in $\tilde{\mathcal{B}}$ for the diagonal action. Then the only $\tilde{v}_0 \in \tilde{V} = W$ such that $\tilde{K}(\tilde{v}_0)$ is closed in $\tilde{\mathcal{B}}$ is $\tilde{v}_0 = 1$, the identity element of W. Let $\tilde{W} = W \times W$, and for $w \in W = \tilde{V}$, let $\tilde{W}'_{\tilde{v}_0}(w) \subset \tilde{W}_{\tilde{v}_0}(w) \subset \tilde{W}$ be defined as in (2.2) and (4.1) but for the pair $(\tilde{G}, \tilde{\theta})$. Then it is easy to see that, for any $w \in W = \tilde{V}$,

$$\tilde{W}_{\tilde{v}_0}(w) = \{(w_1, w_2) \in W \times W : \ w \leq w_1 * w_2^{-1}\},$$

where recall from §3.6 that $*$ is the monoidal product on W, while

$$\tilde{W}'_{\tilde{v}_0}(w) = \{(w_1, w_2) \in W \times W : \ BwB \subset Bw_1Bw_2^{-1}B\},$$

where B is any Borel subgroup of G. The set $\tilde{W}'_{\tilde{v}_0}(w)$ can be computed by choosing any reduced word for w_1 and using inductively the fact that, for $s \in S$ and $u \in W$, $BsBuB = BsuB$ if $su > u$ and $BsBuB = BsuB \cup BuB$ if $su < u$. See [6, Remarques 3.19]. Moreover, for any $w \in W$,

$$\begin{aligned} \min(\tilde{W}_{\tilde{v}_0}(w)) &= \min(\tilde{W}'_{\tilde{v}_0}(w)) \\ &= \{(w_1, w_2) \in W \times W : \ w = w_1 w_2^{-1}, \ l(w) = l(w_1) + l(w_2)\}, \end{aligned}$$

and all elements in $\min(\tilde{W}_{\tilde{v}_0}(w))$ have the same length, namely $l(w)$.

4.4. **The element** $m(w){\cdot}v_0$**.** Fix $v_0 \in V_0$ and let $B \in K(v_0)$. For $w \in W$, let

$$V_w = \{v \in V : \ K(v) \cap B(w) \neq \emptyset\}. \tag{4.4}$$

Then V_w depends only on w and not on the choice of $B \in K(v_0)$.

Proposition 4.7. *Let $v_0 \in V_0$, and let $B \in K(v_0)$. Then for every $w \in W$,*

1) $B(w) \cap K(m(w){\cdot}v_0)$ is dense in $B(w)$,

2) $m(w){\cdot}v_0$ is the unique maximal element in V_w with respect to the Bruhat order on V.

Proof. Since $B(w)$ is irreducible and since $\mathcal{B} = \bigsqcup_{v \in V} K(v)$, there exists a unique $v_w \in V$ such that $B(w) \cap K(v_w)$ is dense in $B(w)$. Moreover, if $v \in V_w$, then

$$\emptyset \neq B(w) \cap K(v) \subset B(w) \subset \overline{B(w)} = \overline{B(w) \cap K(v_w)} \subset \overline{K(v_w)},$$

and so $v \leq v_w$. This shows that v_w is the unique maximal element in V_w with respect to the Bruhat order on V. It remains to show that $v_w = m(w){\cdot}v_0$.

Since $w \in W_{v_0}(v_w)$, $v_w \leq m(w){\cdot}v_0$. By Lemma 3.4, $\overline{K(m(w){\cdot}v_0)} = \overline{B(w) \cdot K}$, so $K(m(w){\cdot}v_0) \cap (B(w) \cdot K) \neq \emptyset$, and hence $K(m(w){\cdot}v_0) \cap B(w) \neq \emptyset$. Thus $m(w){\cdot}v_0 \in V_w$ and $m(w){\cdot}v_0 \leq v_w$. Hence $v_w = m(w){\cdot}v_0$.

Q.E.D.

5. Intersections of G_0-orbits and B-orbits

In this section, we assume that $\mathbf{k} = \mathbb{C}$ and that $G_0 = \{g \in G : \tau(g) = g\}$ is a real form of G, where τ is an anti-holomorphic involution on G. Let σ be a Cartan involution of G commuting with τ, and let $\theta = \sigma\tau$. Let $K = G^\theta$ and $K_0 = G_0 \cap K$. Then K_0 is a maximal compact subgroup of G_0 and K is a complexification of K_0.

If X is a subset of $\mathcal{B}$, $\overline{X}$ in this section will denote the closure of X in $\mathcal{B}$ in the classical topology. If $X = K(v)$ or $X = B(w)$, where $v \in V$, B is any Borel subgroup of G, and $w \in W$, the closure of X in the classical topology coincides with that in the Zariski topology.

5.1. **The Matsuki duality.** By Matsuki duality [8, 10] [13, §6], for each $v \in V$, there exists a unique G_0-orbit $G_0(v)$ in $\mathcal{B}$ such that $G_0(v) \cap K(v)$ is a single K_0-orbit. The map $v \mapsto G_0(v)$ gives an identification of the set $V = \mathcal{B}/K$ of K-orbits in $\mathcal{B}$ with the set $\mathcal{B}/G_0$ of G_0-orbits in $\mathcal{B}$. For $v \in V$, let $K_0(v) = G_0(v) \cap K(v)$.

The following Proposition 5.1 is proved in [10, Corollary 1.4] [13, Theorem 6.4.5].

Proposition 5.1. *For $v, v' \in V$, one has*

$$v \leq v' \quad \text{iff} \quad G_0(v') \subset \overline{G_0(v)} \quad \text{iff} \quad G_0(v) \cap K(v') \neq \emptyset.$$

Let $v_0 \in V_0$ and fix $B \in K(v_0)$. Recall the map $q_B : G \to \mathcal{B} : g \mapsto B^g$ for $g \in G$. For $v \in V$, let

$$BvG_0 = q_B^{-1}(G_0(v)), \quad BvK_0 = q_B^{-1}(K_0(v)),$$

and recall that $BvK = q_B^{-1}(K(v))$. Let $B_K = K \cap B$.

Lemma 5.2. *1) $K = K_0 B_K$ and $G_0 K = G_0 B_K$.*

2) For any $v \in V$, one has $BvG_0K = \bigsqcup_{v' \in V, v' \geq v} Bv'K$.

Proof. 1) Let $U = \{g \in G : \sigma(g) = g\}$ and $H = B \cap \sigma(B)$. Then U is a compact real form of G and H a maximal torus of G. Let $A = \{h \in H : \sigma(h) = h^{-1}\}$ and let N be the uniradical of B. Then one has the Iwasawa decomposition $G = UAN$ of G, and U, A, and N are all θ-invariant. Let $k \in K$ and write $k = uan$ with $u \in U, a \in A$, and $n \in N$. It follows from $\theta(k) = k$ that $u \in K_0$ and $an \in B_K$. Thus $K = K_0 B_K$. It follows from $K_0 \subset G_0$ that $G_0 K = G_0 B_K$.

2) Let $v \in V$. Then BvG_0K is a union of (B, K)-double cosets, and for $v' \in V$,

$$Bv'K \subset BvG_0K \quad \text{iff} \quad Bv'K \cap BvG_0 \neq \emptyset,$$

which, by Proposition 5.1, is equivalent to $v \leq v'$.

Q.E.D.

5.2. **Proof of Theorem 2.1.** Let $v_0 \in V_0$ and let $B \in K(v_0)$. Let $v \in V$ and $w \in W$. We must prove that

$$G_0(v) \cap B(w) \neq \emptyset \quad \text{iff} \quad v \leq m(w) {\cdot} v_0.$$

Let $BwB = q_B^{-1}(B(w)) \subset G$. Then, by definition, $G_0(v) \cap B(w) \neq \emptyset$ if and only if $(BvG_0) \cap (BwB) \neq \emptyset$. Since $B_K \subset B$, this last statement is equivalent to $(BvG_0B_K) \cap (BwB) \neq \emptyset$, which, by 1) of Lemma 5.2, is in turn equivalent to $(BvG_0K) \cap (BwB) \neq \emptyset$. By 2) of Lemma 5.2, $G_0(v) \cap B(w) \neq \emptyset$ if and only if $v \leq v'$ for some $v' \in V_w$, where $V_w = \{v' \in V : \ K(v') \cap B(w) \neq \emptyset\}$.

By Proposition 4.7, $m(w)\cdot v_0$ is the unique maximal element in V_w with respect to the Bruhat order on V. Thus $G_0(v) \cap B(w) \neq \emptyset$ if and only if $v \leq m(w)\cdot v_0$.

This finishes the proof of Theorem 2.1.

Corollary 5.3. *Let $v_0 \in V_0$ and let $B \in K(v_0)$. Then for any $v \in V$ and $w \in W$, the following are equivalent:*

1) $G_0(v) \cap B(w) \neq \emptyset$;
2) $\overline{G_0(v)} \cap B(w) \neq \emptyset$;
3) $G_0(v) \cap \overline{B(w)} \neq \emptyset$;
4) $\overline{G_0(v)} \cap \overline{B(w)} \neq \emptyset$;
5) $K(v) \cap \overline{B(w)} \neq \emptyset$;
6) $K_0(v) \cap \overline{B(w)} \neq \emptyset$.

Proof. Clearly 1) implies 4). Assume 4). Then there exist $v_1 \in V$ and $w_1 \in W$ with $v_1 \geq v$ and $w_1 \leq w$, such that $G_0(v_1) \cap B(w_1) \neq \emptyset$. By Theorem 2.1, $v_1 \leq m(w_1)\cdot v_0$, and by Lemma 3.5, $v \leq v_1 \leq m(w_1)\cdot v_0 \leq m(w)\cdot v_0$. Thus 1) holds by Theorem 2.1. Hence 1) is equivalent to 4) and consequently also equivalent to 2) and 3).

By Theorem 2.1 and Lemma 4.2, both 1) and 5) are equivalent to $v \leq m(w)\cdot v_0$. Since $K = K_0 B_K$ by 1) of Lemma 5.2, 5) and 6) are equivalent.

Q.E.D.

Lemma 5.4. *Let $v_0 \in V_0$ and let $B \in K(v_0)$. Then for any $v \in V$ and $w \in W$, the following are equivalent:*

1) $K(v) \cap B(w) \neq \emptyset$;
2) $K_0(v) \cap B(w) \neq \emptyset$.

Proof. It follows from $K = K_0 B_K$ that 1) and 2) are equivalent.

Q.E.D.

6. Review on K-orbits in $\mathcal{B}$, II

Keeping the notation as in §3, we now review the canonical definitions of more structures on the set V of K-orbits in $\mathcal{B}$, notably the cross action, the Springer map, root types, and reduced decompositions. All the results in this section, except Proposition 6.8 in §6.8, can be found in [12, 13].

6.1. **The canonical maximal torus $H_{\rm can}$ and the W-action on $H_{\rm can}$.** Recall from §3.2 that $\mathcal{C}$ is the variety of all pairs (B, H), where $B \in \mathcal{B}$ and $H \subset B$ is a maximal torus of G. Let $\widetilde{\mathcal{C}} = \{(B, H, h) \in \mathcal{C} \times G : (B, H) \in \mathcal{C},\ h \in H\}$, and let G act on $\widetilde{\mathcal{C}}$ by

$$(B, H, h)^g = (B^g, H^g, g^{-1}hg), \qquad (B, H, h) \in \widetilde{\mathcal{C}},\ g \in G.$$

Let $H_{\mathrm{can}} = \widetilde{\mathcal{C}}/G$ be the set of G-orbits in $\widetilde{\mathcal{C}}$, and let $\widetilde{\mathcal{C}} \to H_{\mathrm{can}} : (B, H, h) \to [B, H, h]$ be the canonical projection. Then for every $(B, H) \in \mathcal{C}$, one has the bijective map

$$T_{B,H}: \; H \longrightarrow H_{\mathrm{can}}: \;\; h \longmapsto [B, H, h], \qquad h \in H.$$

Since $T^{-1}_{B',H'} \circ T_{B,H} = T^{B',H'}_{B,H} : H \to H'$ for any $(B, H), (B', H') \in \mathcal{C}$, where $T^{B',H'}_{B,H}$ is the isomorphism of tori in (3.3), H_{can} has a well-defined structure of a torus, such that $T_{B,H} : H \to H_{\mathrm{can}}$ is an isomorphism of tori for every $(B, H) \in \mathcal{C}$. We will call H_{can} the *canonical maximal torus* of G.

The canonical Weyl group W acts on H_{can} through the action of W_H on H via the identifications $\eta^{-1}_{B,H} : W \to W_H$ and $T^{-1}_{B,H} : H_{\mathrm{can}} \to H$ for any $(B, H) \in \mathcal{C}$, and the action is independent of the choice of $(B, H) \in \mathcal{C}$. If $(B, H), (B', H) \in \mathcal{C}$, then

$$p(B', B)([B, H, h]) = [B', H, h], \qquad h \in H. \tag{6.1}$$

One can also take (6.1) as the definition of the canonical action of W on H_{can}.

6.2. **The canonical root system of G.** Let X_{can} be the character group of H_{can}. For $(B, H) \in \mathcal{C}$, let X_H be the character group of H, and let $R_{B,H} : X_H \to X_{\mathrm{can}}$ be the isomorphism induced by $T^{-1}_{B,H} : H_{\mathrm{can}} \to H$. The *canonical sets of roots, positive roots, and simple roots* of G, are the subsets of X_{can}, respectively defined by

$$\Delta = R_{B,H}(\Delta_H), \qquad \Delta^+ = R_{B,H}(\Delta^+_{B,H}), \qquad \Gamma = R_{B,H}(\Gamma_{B,H}),$$

where $(B, H) \in \mathcal{C}$, and $\Delta_H \supset \Delta^+_{B,H} \supset \Gamma_{B,H}$ are the subsets of the character group X_H of H consisting, respectively, of the roots of (G, H), the positive roots, and the simple roots determined by B. It is easy to check that the sets $\Gamma \subset \Delta^+ \subset \Delta \subset X_{\mathrm{can}}$ are independent of the choice of $(B, H) \in \mathcal{C}$.

For $\alpha \in \Delta$, choose any $(B, H) \in \mathcal{C}$, let $\alpha_H = R^{-1}_{B,H}(\alpha) \in \Delta_H$, let $s_{\alpha_H} \in W_H$ be the reflection defined by α_H, and let $s_\alpha = \eta_{B,H}(s_{\alpha_H}) \in W$. Then s_α is independent of the choice of $(B, H) \in \mathcal{C}$. Moreover, $S = \{s_\alpha : \alpha \in \Gamma\}$.

6.3. **The automorphism θ on H_{can}, Γ, and W.** In this subsection, let θ be any automorphism of G, not necessarily of order 2. Then one has the well-defined map

$$\theta: \; H_{\mathrm{can}} \longrightarrow H_{\mathrm{can}}: \;\; [B, H, h] \longmapsto [\theta(B), \theta(H), \theta(h)], \qquad (B, H, h) \in \widetilde{\mathcal{C}}.$$

If $(B, H) \in \mathcal{C}$, then, by definition, $\theta = T_{\theta(B),\theta(H)} \circ \theta|_H \circ T^{-1}_{B,H}$, where $\theta|_H : H \to \theta(H)$. Thus θ is an automorphism of H_{can}.

It is clear from the definition that if θ is inner, i.e., if there exists $g_1 \in G$ such that $\theta(g) = g_1 g g_1^{-1}$ for $g \in G$, then θ induces the identity automorphism of H_{can}.

We will use the same letter to denote the induced action of θ on the set Δ of canonical roots. It follows from the definitions that $\theta(\Delta^+) = \Delta^+$ and $\theta(\Gamma) = \Gamma$.

The following map, again denoted by θ, is easily seen to be well-defined:

$$\theta: \; W \longrightarrow W: \;\; p(B_1, B_2) \longmapsto p(\theta(B_1), \theta(B_2)), \qquad B_1, B_2 \in \mathcal{B},$$

Choose any $(B, H) \in \mathcal{C}$, and let $\theta_{W_H} : W_H \to W_{\theta(H)}$ be the group isomorphism induced by $\theta|_{N_G(H)} : N_G(H) \to N_G(\theta(H))$. It follows from definitions that

$$\theta = \eta_{\theta(B),\theta(H)} \circ \theta_{W_H} \circ \eta_{B,H}^{-1} : \quad W \longrightarrow W.$$

Thus θ is an automorphism of W. It also follows that $\theta(s_\alpha) = s_{\theta(\alpha)}$ for all $\alpha \in \Delta$.

6.4. **The identification $\gamma : \mathcal{C}_\theta/K \to V$.** Let $\mathcal{T}^\theta$ be the set of all θ-stable maximal tori in G. The following Lemma 6.1 is proved in [13, Proposition 1.2.1] and [16, Corollary 4.4]. See also [12, 1.4(b)].

Lemma 6.1. *Every $B \in \mathcal{B}$ contains some $H \in \mathcal{T}^\theta$, and if $H_1, H_2 \in \mathcal{T}^\theta$ are both contained in B, then $H_1 = k^{-1}H_2k$ for some $k \in B \cap K$.*

Let $\mathcal{C}_\theta = \{(B, H) \in \mathcal{C} : H \in \mathcal{T}^\theta\}$ and let K act on $\mathcal{C}_\theta$ by (3.1). It follows from Lemma 6.1 that the well-defined map

$$\gamma : \ \mathcal{C}_\theta/K \longrightarrow V = \mathcal{B}/K : \ K\text{-orbit of } (B, H) \text{ in } \mathcal{C} \longmapsto K\text{-orbit of } B \text{ in } \mathcal{B},$$

is a bijection ([13, Proposition 1.2.1] and [16, Corollary 4.4]).

6.5. **The cross action of W on V.** We follow [12, §2] and [13, §1.7]. See also [4]. For any $(B, H) \in \mathcal{C}$ and $w \in W$, there is a unique $(B', H) \in \mathcal{C}$ such that $w = p(B', B)$. Define

$$W \times \mathcal{C} \longrightarrow \mathcal{C} : \ \ w \cdot (B, H) = (B', H), \qquad (B, H), (B', H) \in \mathcal{C},\ w = p(B', B). \tag{6.2}$$

If we fix $(B, H) \in \mathcal{C}$, then under the identifications $\eta_{B,H} : W_H \to W$ and $C_{B,H} : H\backslash G \to \mathcal{C} : Hg \mapsto (B^g, H^g)$ for $g \in G$, the map in (6.2) becomes

$$W_H \times H\backslash G \longrightarrow H\backslash G : \quad (nH, Hg) \longmapsto Hng, \qquad n \in N_G(H),\ g \in G,$$

which is a left action of W_H on $H\backslash G$. Thus (6.2) is indeed a left action of W on $\mathcal{C}$. Moreover, the W-action commutes with the right action of G on $\mathcal{C}$ given in (3.1).

By (6.2), $\mathcal{C}_\theta \subset \mathcal{C}$ is W-invariant. Thus one has a well-defined action of W on $\mathcal{C}_\theta/K$. Identifying V with $\mathcal{C}_\theta/K$ via γ, one gets a left action of W on V, which is called the *cross action* of W on V and will be denoted by

$$W \times V \longrightarrow V : \quad (w, v) \longmapsto w{\cdot}v, \qquad w \in W,\ v \in V.$$

6.6. **The Springer map $\phi : V \to W$.** The *Springer map* $\phi : V \to W$, introduced by Springer in [16], is defined (see [12, Remark 1.8] and [13, Proposition 1.7.1]) by

$$\phi(v) = p(B, \theta(B)), \qquad v \in V,\ B \in K(v). \tag{6.3}$$

The element $\phi(v) \in W$ for $v \in V$ is an important invariant of the K-orbit $K(v) \subset \mathcal{B}$. Recall that V_0 consists of all $v \in V$ such that $K(v)$ is closed in $\mathcal{B}$. Let 1 be the identity element of W. The following Proposition 6.2 is from [13, Proposition 1.4.2] and [12, Proposition 2.5].

Proposition 6.2. *1) For $v \in V$, $v \in V_0$ if and only if $\phi(v) = 1$.*

2) If $v, v' \in V$ are such that $\phi(v) = \phi(v')$, then v and v' are in the same W-orbit.

6.7. **Using standard pairs.** Let $(B, H) \in \mathcal{C}$ be a standard pair, i.e., $\theta(B) = B$ and $\theta(H) = H$. Then for any $g \in G$, $(B^g, H^g) \in \mathcal{C}_\theta$ if and only if $g\theta(g)^{-1} \in N_G(H)$, and for such a $g \in G$,

$$p(B^g, \theta(B^g)) = p(B^g, B^{\theta(g)}) = p(B^{g\theta(g)^{-1}}, B) = \eta_{B,H}(g\theta(g)^{-1}H) \in W.$$

Letting $\mathcal{V}_H = \{g \in G : g\theta(g)^{-1} \in N_G(H)\}$, one thus has the identification

$$H\backslash\mathcal{V}_H/K \longrightarrow V : \quad HgK \longmapsto \text{the } K\text{-orbit in } \mathcal{B} \text{ through } B^g. \tag{6.4}$$

Under the identification of V with $H\backslash\mathcal{V}_H/K$ in (6.4) and that of W with W_H by $\eta_{B,H}$, the Springer map $\phi : V \to W$ becomes $\phi(HgK) = g\theta(g)^{-1}H$ for $g \in \mathcal{V}_H$, and the action of W on V becomes $(nH)\cdot(HgK) = HngK$ for $n \in N_G(H)$ and $g \in \mathcal{V}_H$.

In [12, 13, 16], most of the structures on V are introduced and their properties proved by using standard pairs. For example, the following Lemma 6.3 immediately follows from the the identification (6.4) (see also [12, Lemma 2.1]).

Lemma 6.3. *One has $\phi(w\cdot v) = w\phi(v)\theta(w)^{-1}$ for any $w \in W$ and $v \in V$.*

6.8. **The closed K-orbits in $\mathcal{B}$.** Let $W^\theta = \{w \in W : \theta(w) = w\}$. By Lemma 6.3,

$$\{w \in W : w\cdot v_0 = v_0'\} = \{w \in W^\theta : w\cdot v_0 = v_0'\}, \quad \forall\, v_0, v_0' \in V_0. \tag{6.5}$$

By Proposition 6.2, W^θ acts transitively on V_0.

For the rest of this subsection, we assume that K is connected. We will relate the sets in (6.5) for $v_0, v_0' \in V_0$ with the canonical Weyl group W_K of K. Proposition 6.8 will be used in §9.1, where we determine $Y_{v_0}(v_0')$ for every $v_0, v_0' \in V_0$.

Lemma 6.4. *Let $v_0, v_0' \in V_0$. Then for any $B \in K(v_0)$ and $B' \in K(v_0')$, there exists $H \in \mathcal{T}^\theta$ such that $H \subset B \cap B'$.*

Proof. Since $K(v_0)$ and $K(v_0')$ are closed, it follows from [13, Theorem 1.4.3] (see also [16, Corollary 6.6]) that B and B' are θ-stable. Thus, $B \cap B'$ is θ-stable and contains a maximal torus of G. Hence, by [18, Theorem 7.5], $B \cap B'$ contains a θ-stable maximal torus of G.

Q.E.D.

Let $v_0, v_0' \in V_0$, and consider the restriction $p : K(v_0')\times K(v_0) \to W$. By definition,

$$p(K(v_0') \times K(v_0)) = \{w \in W : w = p(B', B) \text{ for some } B' \in K(v_0'),\ B \in K(v_0)\}.$$

Lemma 6.5. *For any $v_0, v_0' \in V_0$, one has*

$$p(K(v_0') \times K(v_0)) = \{w \in W : w\cdot v_0 = v_0'\}.$$

Proof. By the definition of the W-action, $\{w \in W : w{\cdot}v_0 = v_0'\} \subset p(K(v_0') \times K(v_0))$. Suppose that $w = p(B', B)$ for some $B \in K(v_0)$ and $B' \in K(v_0')$. By Lemma 6.4 and by the definition of the W-action on V, $w{\cdot}v_0 = v_0'$.

Q.E.D.

Let $\mathcal{B}_K$ be the variety of all Borel subgroups of K and let K act on $\mathcal{B}_K$ by $(B_K)^k := k^{-1}B_K k$ for $B_K \in \mathcal{B}_K$ and $k \in K$.

Lemma 6.6. *For any $v_0 \in V_0$, the map*

$$\mathcal{I}_{v_0}: \quad K(v_0) \longrightarrow \mathcal{B}_K: \quad B \longmapsto B \cap K, \qquad B \in K(v_0)$$

is a K-equivariant isomorphism.

Proof. By [11, 5.1] (see also [13, Page 113]), $B \cap K \in \mathcal{B}_K$ for every $B \in K(v_0)$. Thus $\mathcal{I}_{v_0}$ is well-defined. It is clear that $\mathcal{I}_{v_0}$ is K-equivariant. To show that $\mathcal{I}_{v_0}$ is an isomorphism, we show that $\mathcal{I}_{v_0}$ is bijective and that its inverse is an isomorphism from $\mathcal{B}_K$ to $K(v_0)$. Fix $B \in K(v_0)$ and identify $(B \cap K)\backslash K \cong \mathcal{B}_K$ via

$$(B \cap K)\backslash K \longrightarrow \mathcal{B}_K: \quad (B \cap K)k \longmapsto (B \cap K)^k, \qquad k \in K.$$

Consider the action map $\eta : K \to K(v_0) : k \mapsto B^k$, $k \in K$. Then the morphism $\tilde{\eta} : (B \cap K)\backslash K \to K(v_0)$ induced by η is the inverse of $\mathcal{I}_{v_0}$. By [5, Proposition 6.7 and Corollary 6.1], $\tilde{\eta}$ is an isomorphism if η is separable.

Let $\mathfrak{b}, \mathfrak{k}$, and $\mathfrak{g}$ be the Lie algebras of B, K, and G respectively, and let $d\theta : \mathfrak{g} \to \mathfrak{g}$ be the differential of θ. Since $\mathrm{char}(\mathbf{k}) \neq 2$, θ is semisimple on G and B [15, Section 5.4]. By [15, Theorem 5.4.4(ii)], $\mathfrak{k} = \mathfrak{g}^{d\theta} = \{x \in \mathfrak{g} : d\theta(x) = x\}$, and the Lie algebra of $B \cap K$ coincides with $\mathfrak{b} \cap \mathfrak{k}$. Applying [5, Proposition 6.12] to the quotient morphism $G \to B\backslash G : g \to B^g, g \in G$, one sees that η is separable.

Q.E.D.

Identify $W_K = (\mathcal{B}_K \times \mathcal{B}_K)/K$ for the diagonal action of K on $\mathcal{B}_K \times \mathcal{B}_K$. For $x \in W_K$, let $\mathcal{O}_K(x) \subset \mathcal{B}_K \times \mathcal{B}_K$ be the corresponding K-orbit in $\mathcal{B}_K \times \mathcal{B}_K$. Let $\leq_K$ denote the Bruhat order on W_K.

Let $v_0, v_0' \in V_0$, and let

$$\mathcal{J}_{v_0,v_0'} = (\mathcal{I}_{v_0'} \times \mathcal{I}_{v_0})^{-1}: \; \mathcal{B}_K \times \mathcal{B}_K \longrightarrow K(v_0') \times K(v_0).$$

For $x \in W_K$, let $\mathcal{O}_{K,v_0,v_0'}(x)$ be the single K-orbit in $K(v_0') \times K(v_0)$ given by

$$\mathcal{O}_{K,v_0,v_0'}(x) = \mathcal{J}_{v_0,v_0'}(\mathcal{O}_K(x)) \subset K(v_0') \times K(v_0). \tag{6.6}$$

One then has the well-defined map

$$I_{v_0,v_0'}: \quad W_K \longrightarrow W: \quad x \longmapsto p\left(\mathcal{O}_{K,v_0,v_0'}(x)\right), \qquad x \in W_K. \tag{6.7}$$

Let 1 be the identity element in W_K. By definition,

$$\mathcal{O}_{K,v_0,v_0'}(1) = \{(B', B) \in K(v_0') \times K(v_0): \; B' \cap K = B \cap K\}.$$

Definition 6.7. For $v_0, v_0' \in V_0$, let $y_{v_0,v_0'} \in W^\theta$ be given by

$$y_{v_0,v_0'} = I_{v_0,v_0'}(1) = p(B', B), \quad \forall (B', B) \in \mathcal{O}_{K,v_0,v_0'}(1). \tag{6.8}$$

Proposition 6.8. *Let $v_0, v_0' \in V_0$.*

1) The map $I_{v_0,v_0'}$ is a bijection from W_K onto $\{w \in W^\theta : w\cdot v_0 = v_0'\}$;

2) If $x_1, x_2 \in W_K$ are such that $x_1 \leq_K x_2$, then $I_{v_0,v_0'}(x_1) \leq I_{v_0,v_0'}(x_2)$;

3) $y_{v_0,v_0'} \in W^\theta$ is the unique minimal element in the set $\{w \in W^\theta : w\cdot v_0 = v_0'\}$ with respect to the Bruhat order on W.

Proof. 1) By Lemma 6.5 and (6.5), the image of $I_{v_0,v_0'}$ is $\{w \in W^\theta : w\cdot v_0 = v_0'\}$. To show that $I_{v_0,v_0'}$ is injective, assume that $B, B_1 \in K(v_0)$ and $B', B_1' \in K(v_0')$ are such that $p(B', B) = p(B_1', B_1) \in W$. We must show that (B', B) and (B_1', B_1) are in the same K-orbit in $K(v_0') \times K(v_0)$ for the diagonal action of K. Without loss of generality, we may assume that $B_1 = B$, and we need to show that $B' = (B_1')^k$ for some $k \in B \cap K$. Let $H, H_1 \in \mathcal{T}^\theta$ be such that $H \subset B \cap B'$ and $H_1 \subset B \cap B_1'$. By Lemma 6.1, there exists $k \in B \cap K$ such that $H = (H_1)^k$. Thus $H \subset B \cap B' \cap (B_1')^k$. By the assumption, $p(B', B) = p(B_1', B) = p((B_1')^k, B^k) = p((B_1')^k, B)$. Thus $B' = (B_1')^k$.

2) Suppose that $x_1, x_2 \in W_K$ are such that $x_1 \leq_K x_2$. For $i = 1, 2$, let $w_i = I_{v_0,v_0'}(x_i) \in W$, so that $\mathcal{O}_{K,v_0,v_0'}(x_i) \subset \mathcal{O}(w_i)$, where recall from §3.4 that $\mathcal{O}(w_i)$ is the G-orbit in $\mathcal{B} \times \mathcal{B}$ corresponding to $w_i \in W$. For a subset X of $\mathcal{B}_K \times \mathcal{B}_K$ (resp. of $\mathcal{B} \times \mathcal{B}$), let $\overline{X}^{\mathcal{B}_K \times \mathcal{B}_K}$ (resp. $\overline{X}^{\mathcal{B} \times \mathcal{B}}$) denote the Zariski closure of X in $\mathcal{B}_K \times \mathcal{B}_K$ (resp. in $\mathcal{B} \times \mathcal{B}$). Since $\mathcal{J}_{v_0,v_0'} : \mathcal{B}_K \times \mathcal{B}_K \to \mathcal{B} \times \mathcal{B}$ is a morphism, one has

$$\begin{aligned}\mathcal{O}_{K,v_0,v_0'}(x_1) = \mathcal{J}_{v_0,v_0'}(\mathcal{O}_K(x_1)) &\subset \mathcal{J}_{v_0,v_0'}\left(\overline{\mathcal{O}_K(x_1)}^{\mathcal{B}_K \times \mathcal{B}_K}\right) \subset \overline{\mathcal{J}_{v_0,v_0'}(\mathcal{O}_K(x_1))}^{\mathcal{B} \times \mathcal{B}} \\ &\subset \overline{\mathcal{O}(w_2)}^{\mathcal{B} \times \mathcal{B}}.\end{aligned}$$

Thus $\mathcal{O}(w_1) \cap \overline{\mathcal{O}(w_2)}^{\mathcal{B} \times \mathcal{B}} \neq \emptyset$, and hence $w_1 \leq w_2$.

3) follows directly from 1) and 2).

Q.E.D.

Remark 6.9. For $v_0 \in V_0$, let $W_K(v_0) = \{w \in W : wv_0 = v_0\}$. Then for $v_0, v_0' \in V_0$, the set $\{w \in W^\theta : w\cdot v_0 = v_0'\}$ coincides with the coset $y_{v_0,v_0'} W_K(v_0)$ in W^θ. It is easy to see that $I_{v_0,v_0} : W_K \to W_K(v_0)$ is a group isomorphism (see, for example, [12, Proposition 2.8]). Hence, by 3) of Proposition 6.8, every coset in $W^\theta / W_K(v_0)$ has a unique minimal length representative. In case K is disconnected, this last assertion is no longer true in the case when $G = PGL(4)$ and K has connected component of the identity equal to the image of $GL(2) \times GL(2)$ in G.

6.9. **The involution $\theta_v : H_{\rm can} \to H_{\rm can}$.** Let $v \in V$. Choose any $(B, H) \in \mathcal{C}_\theta$ such that $B \in K(v)$, and let $\theta|_H : H \to H$ be the restriction of θ to H. Define

$$\theta_v := T_{B,H} \circ \theta|_H \circ T_{B,H}^{-1} : \ H_{\rm can} \longrightarrow H_{\rm can} : \ \ [B, H, h] \longmapsto [B, H, \theta(h)], \quad h \in H.$$

For another $(B', H') \in \mathcal{C}_\theta$ such that $B' \in K(v)$, there exists $k \in K$ such that $B' = B^k$ and $H' = H^k$, and for any $h \in H$, $[B, H, h] = [B^k, H^k, k^{-1}hk] \in H_{\rm can}$, and

$$[B^k, H^k, \theta(k^{-1}hk)] = [B^k, H^k, k^{-1}\theta(h)k] = [B, H, \theta(h)].$$

Thus $\theta_v : H_{\rm can} \to H_{\rm can}$ is independent of the choice of $(B, H) \in \mathcal{C}_\theta$. By (6.1),

$$\theta_v = \phi(v)\theta : \quad H_{\rm can} \longrightarrow H_{\rm can}. \tag{6.9}$$

The induced involution on the set Δ of canonical roots of G (see §6.2) will also be denoted by θ_v.

6.10. **The subsets $p(s_\alpha, v)$ and root types.** Let $\alpha \in \Gamma$. A parabolic subgroup of G is said to be of *type* α if it is of the form $B \cup Bs_\alpha B$ for some $B \in \mathcal{B}$. Let $\mathcal{P}_\alpha$ be the variety of all parabolic subgroups of G of type α. The action of G on $\mathcal{P}_\alpha$ by conjugation is transitive, and one has the G-equivariant surjective morphism

$$\pi_\alpha : \ \mathcal{B} \longrightarrow \mathcal{P}_\alpha : \ \ \pi_\alpha(B) = B \cup (Bs_\alpha B), \qquad B \in \mathcal{B}.$$

For $v \in V$, let

$$p(s_\alpha, v) = \{v' \in V : \ K(v') \subset \pi_\alpha^{-1}(\pi_\alpha(K(v))\}.$$

It is well-known (see, for example, [13, §2.4] and [14]) that for each $\alpha \in \Gamma$ and $v \in V$, the subset $p(s_\alpha, v)$ of V has either one or two or three elements, depending on the *type* of α relative to v, and that $m(s_\alpha) \cdot v$ is the unique maximal element in $p(s_\alpha, v)$ with respect to the Bruhat order on V.

The case analysis of $p(s_\alpha, v)$ and the definition of the type of α relative to v given in [13, §2.4] and [14] make use of a standard pair $(B_0, H_0) \in \mathcal{C}$, but the results are independent of the choice of (B_0, H_0). Based on the results from [13, §2.4] and [14], we give the equivalent definitions of root types in Definition 6.10 and summarize the results on $p(s_\alpha, v)$ from [13, §2.4] and [14] in the following Proposition 6.11.

Definition 6.10. Let $v \in V$. An $\alpha \in \Gamma$ is said to be *imaginary* (resp. *real*, *complex*) for v if $\theta_v(\alpha) = \alpha$ (resp. $\theta_v(\alpha) = -\alpha$, $\theta_v(\alpha) \neq \pm\alpha$). A simple imaginary root α for $v \in V$ is said to be *compact* if $m(s_\alpha) \cdot v = v$ and *non-compact* if $m(s_\alpha) \cdot v \neq v$. A simple non-compact imaginary root α is said to be *cancellative* if $s_\alpha \cdot v = v$ and *non-cancellative* if $s_\alpha \cdot v \neq v$ (see [17, §2.4]). A simple real root α for $v \in V$ is said to be *cancellative* if $p(s_\alpha, v)$ has two elements and *non-cancellative* if $p(s_\alpha, v)$ has three elements. We will use the following notation.

$$\begin{aligned}
I_v^c &= \{\alpha\in\Gamma:\ \alpha \text{ is compact imaginary for } v\},\\
I_v^{n,=} &= \{\alpha\in\Gamma:\ \alpha \text{ is non-compact imaginary and cancellative for } v\},\\
I_v^{n,\neq} &= \{\alpha\in\Gamma:\ \alpha \text{ is non-compact imaginary and non-cancellative for } v\},\\
R_v^{=} &= \{\alpha\in\Gamma:\ \alpha \text{ is real and cancellative for } v\},\\
R_v^{\neq} &= \{\alpha\in\Gamma:\ \alpha \text{ is real and non-cancellative for } v\},\\
C_v^{+} &= \{\alpha\in\Gamma:\ \alpha \text{ is complex for } v \text{ and } \theta_v(\alpha)\in\Delta^+\},\\
C_v^{-} &= \{\alpha\in\Gamma:\ \alpha \text{ is complex for } v \text{ and } \theta_v(\alpha)\in-\Delta^+\}.
\end{aligned}$$

We also set $I_v^n = I_v^{n,=}\cup I_v^{n,\neq}$, $I_v = I_v^c\cup I_v^n$, and $R_v = R_v^{=}\cup R_v^{\neq}$.

Proposition 6.11. *Let $v\in V$ and $\alpha\in\Gamma$.*

Case 1), $\alpha\in I_v^c$. Then $p(s_\alpha, v)=\{v\}$.

Case 2), $\alpha\in I_v^{n,=}$. Then $s_\alpha\cdot v = v\neq m(s_\alpha)\cdot v$, and $p(s_\alpha, v)=\{v,\ m(s_\alpha)\cdot v\}$. Moreover, $\alpha\in R^{=}_{m(s_\alpha)\cdot v}$.

Case 3), $\alpha\in I_v^{n,\neq}$. Then v, $s_\alpha\cdot v$, and $m(s_\alpha)\cdot v$ are pair-wise distinct, and $p(s_\alpha,v)=\{v,\ s_\alpha\cdot v,\ m(s_\alpha)\cdot v\}$. Moreover, $\alpha\in I^{n,\neq}_{s_\alpha\cdot v}$ and $\alpha\in R^{\neq}_{m(s_\alpha)\cdot v}$.

Case 4), $\alpha\in R_v^{=}$. Then there exists $v'\in p(s_\alpha,v)$, $v'\neq v$, such that $s_\alpha\cdot v'=v'$, $v=m(s_\alpha)\cdot v' = s_\alpha\cdot v$, and $p(s_\alpha,v)=\{v',\ v\}$. Moreover, $\alpha\in I^{n,=}_{v'}$.

Case 5), $\alpha\in R_v^{\neq}$. Then there exists $v'\in p(s_\alpha,v)$ such that v', $s_\alpha\cdot v'$ and $v=m(s_\alpha)\cdot v' = s_\alpha\cdot v$ are pair-wise distinct, and $p(s_\alpha,v)=\{v',\ s_\alpha\cdot v',\ v\}$. Moreover, $\alpha\in I^{n,\neq}_{v'}\cap I^{n,\neq}_{s_\alpha\cdot v'}$.

Case 6), $\alpha\in C_v^+$. Then $m(s_\alpha)\cdot v = s_\alpha\cdot v\neq v$, and $p(s_\alpha,v)=\{v,\ m(s_\alpha)\cdot v\}$. Moreover, $\alpha\in C^-_{m(s_\alpha)\cdot v}$.

Case 7), $\alpha\in C_v^-$. Then $m(s_\alpha)\cdot v = v\neq s_\alpha\cdot v$, and $p(s_\alpha,v)=\{v,\ s_\alpha\cdot v\}$. Moreover, $\alpha\in C^+_{s_\alpha\cdot v}$.

Lemma 6.12. [13, Page 122] *Let $v, v'\in V$ and $\alpha\in\Gamma$. Then $v'\in p(s_\alpha,v)$ if and only if $m(s_\alpha)\cdot v' = m(s_\alpha)\cdot v$. Moreover, $p(s_\alpha,v')=p(s_\alpha,v)$ for all $v'\in p(s_\alpha,v)$.*

Lemma 6.13. [13, §3.2] *For $\alpha\in\Gamma$ and $v\in V$, $m(s_\alpha)\cdot v\neq v$ if and only if $\alpha\in C_v^+\cup I_v^n$. If $\alpha\in C_v^+$, then $\phi(m(s_\alpha)\cdot v)=s_\alpha\phi(v)s_{\theta(\alpha)}$, and $l(\phi(m(s_\alpha)\cdot v)) = l(\phi(v))+2$. If $\alpha\in I_v^n$, then $\phi(m(s_\alpha)\cdot v)=s_\alpha\phi(v)=\phi(v)s_{\theta(\alpha)}>\phi(v)$ and $\phi(s_\alpha\cdot v)=\phi(v)$.*

6.11. **Reduced decompositions and subexpressions.** We refer to [12, §5-7] and [13, §4] for more detail on this subsection.

Definition 6.14. [13, Definition 3.2.3] Let $v\in V$. A *reduced decomposition* of v is a pair $(\mathbf{v},\mathbf{s})$, where $\mathbf{v}=(v_0,v_1,\ldots,v_k)$ is a sequence in V and $\mathbf{s}=(s_{\alpha_1},\ldots,s_{\alpha_k})$ is a sequence of simple reflections in W, such that $v_0\in V_0$, $v_k=v$, and for each $j\in[1,k]$,

$$\alpha_j\in C^+_{v_{j-1}}\cup I^n_{v_{j-1}} \quad\text{and}\quad v_j = m(s_{\alpha_j})\cdot v_{j-1}.$$

The integer k is called the length of the reduced decomposition $(\mathbf{v}, \mathbf{s})$.

Every $v \in V$ has a reduced decomposition and all reduced decompositions of v have the same length, which will be denoted by $l(v)$ and called the *length* of v (see [13, 3.2 and Page 113]).

Definition 6.15. [13, Definition 4.3] Let $v \in V$ and let $(\mathbf{v} = (v_0, v_1, \ldots, v_k), \mathbf{s} = (s_{\alpha_1}, \ldots, s_{\alpha_k}))$ be a reduced decomposition for v. A *subexpression* of $(\mathbf{v}, \mathbf{s})$ is a sequence $\mathbf{u} = (u_0, u_1, \ldots, u_k)$ in V such that $u_0 = v_0$ and one of the following holds for each $j \in [1, k]$:

Case 1), $u_j = u_{j-1}$;

Case 2), $\alpha_j \in C^+_{u_{j-1}} \cup I^n_{u_{j-1}}$ and $u_j = m(s_{\alpha_j}) \cdot u_{j-1}$;

Case 3), $\alpha_j \in I^{n,\neq}_{u_{j-1}}$ and $u_j = s_{\alpha_j} \cdot u_{j-1}$.

In this case, u_k is called the *final term* of the subexpression $\mathbf{u} = (u_0, u_1, \ldots, u_k)$.

Proposition 6.16. [13, Proposition 4.4] *Let $v, v' \in V$ and let $(\mathbf{v}, \mathbf{s})$ be a reduced decomposition for v. Then $v' \leq v$ if and only if there exists a subexpression of $(\mathbf{v}, \mathbf{s})$ with final term v'.*

7. Analysis on $Y_{v_0}(v)$ and proof of Theorem 2.2

7.1. **The set Y_{v_0}.** For $v_0 \in V_0$, let $Y_{v_0} = \bigcup_{v \in V} Y_{v_0}(v)$. Recall that l denotes both the length function on W and the length function on V.

Lemma 7.1. *Let $v_0 \in V_0$ and $y \in W$. Then $y \in Y_{v_0}$ if and only if $l(m(y) \cdot v_0) = l(y)$.*

Proof. It is clear that $l(m(y) \cdot v_0) \leq l(y)$ for any $y \in W$. Assume first that $y \in Y_{v_0}(v)$ for some $v \in V$. If $l(m(y) \cdot v_0) < l(y)$, then there exists $y_1 < y$ such that $m(y_1) \cdot v_0 = m(y) \cdot v_0$, so $y_1 \in W_{v_0}(v)$, which is a contradiction. Thus $l(m(y) \cdot v_0) = l(y)$. Conversely, assume that $l(m(y) \cdot v_0) = l(y)$. Let $v = m(y) \cdot v_0$. Then $y \in Y_{v_0}(v)$.

Q.E.D.

Lemma-Notation 7.2. *Let $y \in W$. If $y = s_{\alpha_k} \cdots s_{\alpha_1}$ is a reduced word of y, the sequence $\mathbf{s} = (s_{\alpha_1}, \ldots, s_{\alpha_k})$ is also called a reduced word of y^{-1}. Let*

$$R(y^{-1}) = \{\mathbf{s} : \ \mathbf{s} \text{ is a reduced word of } y^{-1}\}.$$

Let $v_0 \in V_0$ and $y \in Y_{v_0}$. For $\mathbf{s} = (s_{\alpha_1}, \ldots, s_{\alpha_k}) \in R(y^{-1})$, let

$$\mathbf{v}_{v_0}(\mathbf{s}) = (v_0, \ m(y_1) \cdot v_0, \ \cdots, \ m(y_k) \cdot v_0), \tag{7.1}$$

where $y_j = s_{\alpha_j} \cdots s_{\alpha_1}$ for $j \in [1, k]$. Then $(\mathbf{v}_{v_0}(\mathbf{s}), \mathbf{s})$ is a reduced decomposition for $m(y) \cdot v_0$, which will be called the reduced decomposition of $m(y) \cdot v_0$ associated to $\mathbf{s}$.

Proof. If $(\mathbf{v}_{v_0}(\mathbf{s}), \mathbf{s})$ is not a reduced decomposition for $m(y) \cdot v_0$, then $l(m(y) \cdot v_0) < l(y)$, and so $y \notin Y_{v_0}$ by Lemma 7.1.

Q.E.D.

7.2. **Local analysis of $Y_{v_0}(v)$, Part I.** Recall the monoidal operation $*$ on W in §3.6.

Lemma 7.3. *Let $v_0 \in V_0$, $v \in V$, and $\alpha \in \Gamma$.*

1) If $w \in W_{v_0}(m(s_\alpha)\cdot v)$, then $w \in W_{v_0}(v')$ for all $v' \in p(s_\alpha, v)$;

2) If $w \in W_{v_0}(v)$, then $s_\alpha * w \in W_{v_0}(v')$ for all $v' \in p(s_\alpha, v)$.

Proof. 1) follows from the fact that $v' \leq m(s_\alpha)\cdot v$ for all $v' \in p(s_\alpha, v)$.

2). Assume that $w \in W_{v_0}(v)$. Then by Lemma 3.5,

$$m(s_\alpha)\cdot v \leq m(s_\alpha)\cdot m(w)\cdot v_0 = m(s_\alpha * w)\cdot v_0.$$

Thus $s_\alpha * w \in W_{v_0}(m(s_\alpha)\cdot v)$. 2) now follows from 1).

Q.E.D.

The following Lemma 7.4 is the key in proving Theorem 2.2.

Lemma 7.4. *Let $v_0 \in V_0$, $v \in V$, and $y \in Y_{v_0}(v)$. Let $\alpha \in \Gamma$ and $y' \in W$ be such that $y = s_\alpha y' > y'$. Then there exists $u \in p(s_\alpha, v)\backslash\{v, m(s_\alpha)\cdot v\}$ such that $y' \in Y_{v_0}(u)$. Moreover, there are three possibilities:*

1) $\alpha \in C_u^+$ and $v = m(s_\alpha)\cdot u$. In this case, $\alpha \in C_v^-$;

2) $\alpha \in I_v^n$ and $v = m(s_\alpha)\cdot u$. In this case, $\alpha \in R_v$;

3) $\alpha \in I_u^{n,\neq}$ and $v = s_\alpha\cdot u$. In this case, $\alpha \in I_v^{n,\neq}$.

Proof. Let $y' = s_{\alpha_{k-1}} \cdots s_{\alpha_1}$ be a reduced word of y'. Then

$$\mathbf{s} = (s_{\alpha_1}, \ldots, s_{\alpha_{k-1}}, s_{\alpha_k}) \in R(y^{-1}),$$

where $\alpha_k = \alpha$. Consider the reduced decomposition $(\mathbf{v}_{v_0}(\mathbf{s}), \mathbf{s})$ of $m(y)\cdot v_0$. By Proposition 6.16, there is a subexpression $\mathbf{u} = (u_0 = v_0, u_1, , \ldots, , u_{k-1}, u_k)$ of $(\mathbf{v}_{v_0}(\mathbf{s}), \mathbf{s})$ with final term $u_k = v$, and we let $u = u_{k-1}$. Since $y' < y$ and $u \leq m(y')v_0$ by Proposition 6.16, $u \neq v$ and $u \neq m(s_\alpha)\cdot u = m(s_\alpha)\cdot v$. By the definition of a subexpression, one has either 1), 2), or 3). By Proposition 6.11, $\alpha \in C_v^-$ in 1), $\alpha \in R_v$ in 2), and $\alpha \in I_v^{n,\neq}$ in 3).

It remains to show that $y' \in Y_{v_0}(u)$. Since $u \leq m(y')\cdot v_0$, $y' \in W_{v_0}(u)$. Suppose that $y' \notin Y_{v_0}(u)$. Let $y'' \in W_{v_0}(u)$ be such that $y'' < y'$. By Lemma 7.3, $s_\alpha * y'' \in W_{v_0}(v)$. Since $s_\alpha * y'' \leq s_\alpha * y' = y$ and $l(s_\alpha * y'') \leq 1 + l(y'') < 1 + l(y') = l(y)$, we have $s_\alpha * y'' < y$, which is a contradiction. Thus $y' \in Y_{v_0}(u)$.

Q.E.D.

7.3. **Proof of Theorem 2.2.** Assume that $v, v' \in V$ are such that $\phi(v) = \phi(v')$ and that there exists $y \in Y_{v_0}(v) \cap Y_{v_0}(v')$. We use induction on $l(y)$ to show that $v = v'$.

If $l(y) = 0$, then $y = 1$. By Lemma 4.4, $v = v' = v_0$. Assume now that $l(y) \geq 1$, and choose $\alpha \in \Gamma$ such that $y' := s_\alpha y < y$. By Lemma 7.4, there exist $u \in p(s_\alpha, v)$ and $u' \in p(s_\alpha, v')$ such that $y' \in Y_{v_0}(u) \cap Y_{v_0}(u')$, and

$$\alpha \in \left(C_v^- \cup R_v \cup I_v^{n,\neq}\right) \cap \left(C_{v'}^- \cup R_{v'} \cup I_{v'}^{n,\neq}\right).$$

Since $\phi(v) = \phi(v')$, one has by (6.9) that $\theta_v(\alpha) = \theta_{v'}(\alpha)$. Thus $\alpha \in C_v^-$ (resp. R_v, $I_v^{n,\neq}$) if and only if $\alpha \in C_{v'}^-$ (resp. $R_{v'}$, $I_{v'}^{n,\neq}$). We now look at the cases separately.

Case 1): $\alpha \in C_v^-$. In this case, $\alpha \in C_{v'}^-$, $v = m(s_\alpha)\cdot u = s_\alpha \cdot u$, and $v' = m(s_\alpha)\cdot u' = s_\alpha \cdot u'$. By Lemma 6.13, $\phi(u) = s_\alpha \phi(v)\theta(s_\alpha) = s_\alpha \phi(v')\theta(s_\alpha) = \phi(u')$. By the induction assumption, $u = u'$. Thus $v = s_\alpha \cdot u = s_\alpha \cdot u' = v'$.

Case 2): $\alpha \in R_v$. In this case, $\alpha \in R_{v'}$, $v = m(s_\alpha)\cdot u$, and $v' = m(s_\alpha)\cdot u'$. By Lemma 6.13, $\phi(u) = s_\alpha \phi(v) = s_\alpha \phi(v') = \phi(u')$. By the induction assumption, $u = u'$. Thus $v = m(s_\alpha)\cdot u = m(s_\alpha)\cdot u' = v'$.

Case 3): $\alpha \in I_v^{n,\neq}$. In this case, $\alpha \in I_{v'}^{n,\neq}$, $u = s_\alpha \cdot v$, and $u' = s_\alpha \cdot v'$. By Lemma 6.13, $\phi(u) = \phi(v) = \phi(v') = \phi(u')$. By the induction assumption, $u = u'$. Thus $v = s_\alpha \cdot u = s_\alpha \cdot u' = v'$.

This finishes the proof of Theorem 2.2.

7.4. **Local analysis on $Y_{v_0}(v)$, Part II.** The following Lemma 7.5 strengthens Lemma 7.4 and completes the local analysis on the sets $Y_{v_0}(v)$.

Lemma 7.5. *The element u in Lemma 7.4 is unique. If 2) of Lemma 7.4 occurs and if $\alpha \in I_u^{n,\neq}$, then $y \in Y_{v_0}(s_\alpha \cdot u)$. If 3) of Lemma 7.4 occurs, then $y \in Y_{v_0}(m(s_\alpha)\cdot v)$.*

Proof. The only case where u might not be unique is when 2) in Lemma 7.4 occurs and when $\alpha \in I_u^{n,\neq}$. Assume this is the case. Since $\phi(u) = \phi(s_\alpha \cdot u)$, by Theorem 2.2, y' can not be in both $Y_{v_0}(u)$ and $Y_{v_0}(s_\alpha \cdot u)$, so the choice of u is unique. Moreover, by 1) of Lemma 7.3, $y \in W_{v_0}(s_\alpha \cdot u)$. Let $y_1 \in Y_{v_0}(s_\alpha \cdot u)$ be such that $y_1 \leq y$. By 2) of Lemma 7.3, $s_\alpha * y_1 \in W_{v_0}(v)$. Since $s_\alpha * y_1 \leq s_\alpha * y = y$, one has $s_\alpha * y_1 = y$. If $s_\alpha y_1 > y_1$, then $s_\alpha y_1 = s_\alpha * y_1 = y$, so $y_1 = s_\alpha y = y'$, contradicting the fact that y' can not be in both $Y_{v_0}(u)$ and $Y_{v_0}(s_\alpha \cdot u)$. Thus $s_\alpha y_1 < y_1$, and hence $y = s_\alpha * y_1 = y_1 \in Y_{v_0}(s_\alpha \cdot u)$.

Assume now that 3) of Lemma 7.4 occurs. Then $y = s_\alpha y' \in W_{v_0}(m(s_\alpha)\cdot v)$ by 2) of Lemma 7.3. Let $y_2 \in Y_{v_0}(m(s_\alpha)\cdot v)$ be such that $y_2 \leq y$. By 1) of Lemma 7.3, $y_2 \in W_{v_0}(v)$. Thus $y = y_2 \in Y_{v_0}(m(s_\alpha)\cdot v)$.

Q.E.D.

7.5. **Elements of $Y_{v_0}(v)$ and subexpressions.** We now prove the following key property of $y \in Y_{v_0}(v)$ in terms of subexpressions of reduced decompositions of $m(y)\cdot v_0$.

Proposition 7.6. *Let $v_0 \in V_0, v \in V$ and $y \in Y_{v_0}(v)$. Then for any reduced word $\mathbf{s} = (s_{\alpha_1}, \ldots, s_{\alpha_k})$ of y^{-1}, there is exactly one subexpression $\mathbf{u} = (v_0, u_1, \ldots, u_k)$ of the reduced decomposition $(\mathbf{v}_{v_0}(\mathbf{s}), \mathbf{s})$ of $m(y) \cdot v_0$ that has final term v. Moreover, for any $1 \leq j \leq k$, $u_j \neq u_{j-1}$ and $s_{\alpha_j} s_{\alpha_{j-1}} \cdots s_{\alpha_1} \in Y_{v_0}(u_j)$.*

Proof. By Proposition 6.16, there is a subexpression $\mathbf{u} = (u_0 = v_0, u_1, \ldots, u_{k-1}, u_k)$ of the reduced decomposition $(\mathbf{v}_{v_0}(\mathbf{s}), \mathbf{s})$ of $m(y) \cdot v_0$ with final term v. Letting $\alpha = \alpha_k$ and $y' = s_{\alpha_{k-1}} \cdots s_{\alpha_1}$, one knows from Lemma 7.4 and Lemma 7.5 that $y' \in Y_{v_0}(u_{k-1})$ and that u_{k-1} is uniquely determined by the quadruple (v_0, v, y, α). Proceeding inductively, one proves Proposition 7.6.

Q.E.D.

Notation 7.7. For $v_0 \in V_0$, $v \in V$, $y \in Y_{v_0}(v)$, and $\mathbf{s} \in R(y^{-1})$, the unique subexpression of the reduced decomposition $(\mathbf{v}_{v_0}(\mathbf{s}), \mathbf{s})$ of $m(y) \cdot v_0$ (see Lemma-Notation 7.2) with final term v will be denoted by $\mathbf{u}_{v_0,v}(\mathbf{s})$.

8. Admissible paths and the sets $Y_{v_0}(v)$ and $Z_{v_0}(v)$

8.1. **Admissible paths.** Proposition 7.6 leads naturally to the following notion of admissible paths.

Definition 8.1. Fix $v_0 \in V_0$.

i) For $v \in V$, an *admissible path from* v_0 *to* v is a pair $(\mathbf{v}, \mathbf{s})$, where $\mathbf{v} = (v_0, v_1, \ldots, v_k = v)$ is a sequence in V, and $\mathbf{s} = (s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_k})$ is a sequence of simple reflections in W, and $k \geq 1$, such that one of the following holds for each $j \in [1, k]$:

1) $\alpha_j \in C^+_{v_{j-1}} \cup I^n_{v_{j-1}}$ and $v_j = m(s_{\alpha_j}) \cdot v_{j-1}$;

2) $\alpha_j \in I^{n,\neq}_{v_{j-1}}$ and $v_j = s_{\alpha_j} \cdot v_{j-1}$.

The pair $(\mathbf{v} = \{v_0\}, \mathbf{s} = \emptyset)$ is called the trivial admissible path from v_0 to v_0 with the understanding that $k = 0$ in this case. The set of all admissible paths from v_0 to v is denoted by $\mathcal{P}(v_0, v)$.

ii) For $(\mathbf{v}, \mathbf{s}) \in \mathcal{P}(v_0, v)$ as in i), the number k is called the *length* of $(\mathbf{v}, \mathbf{s})$ and denoted by $l(\mathbf{v}, \mathbf{s})$. We also set $y_0(\mathbf{v}, \mathbf{s}) = 1 \in W$, $y_j(\mathbf{v}, \mathbf{s}) = s_{\alpha_j} s_{\alpha_{j-1}} \cdots s_{\alpha_1}$ for $j \in [1, k]$, and $y(\mathbf{v}, \mathbf{s}) = y_k(\mathbf{v}, \mathbf{s})$.

iii) For $v \in V$, $(\mathbf{v}, \mathbf{s}) \in \mathcal{P}(v_0, v)$ is said to be a *shortest admissible path* from v_0 to v if $l(\mathbf{v}, \mathbf{s}) \leq l(\mathbf{v}', \mathbf{s}')$ for every $(\mathbf{v}', \mathbf{s}') \in \mathcal{P}(v_0, v)$. The length of a shortest admissible path from v_0 to v will be denoted by $l_{v_0}(v)$.

Remark 8.2. We are using the symbol $(\mathbf{v}, \mathbf{s})$ to denote both a reduced decomposition of an element in V and an admissible path in V. This should cause no confusion as we will always use the modifiers "reduced decomposition" or "admissible path" in front of $(\mathbf{v}, \mathbf{s})$. Moreover, it is clear from the definitions that a reduced decomposition of $v \in V$ starting from $v_0 \in V_0$ is an admissible path from v_0 to v.

Lemma 8.3. *Let* $v_0 \in V_0$ *and* $v \in V$. *Let* $(\mathbf{v}, \mathbf{s}) \in \mathcal{P}(v_0, v)$, *where* $\mathbf{s} = (s_1, \ldots s_k)$. *Then* $v \leq m(s_k * \cdots * s_1) \cdot v_0$.

Proof. Let $\mathbf{v} = (v_0, v_1 \ldots, v_{k-1}, v_k = v)$. Then

$$v \leq m(s_k) \cdot v_{k-1} \leq \cdots \leq m(s_k) m(s_{k-1}) \cdots m(s_1) \cdot v_0 = m(s_k * \cdots * s_1) \cdot v_0.$$

Q.E.D.

8.2. **Minimal admissible paths and elements in $Y_{v_0}(v)$.** By Definition 6.15, for $v_0 \in V_0, v \in V, y \in Y_{v_0}(v)$, and $\mathbf{s} \in R(y^{-1})$, the pair $(\mathbf{u}_{v_0,v}(\mathbf{s}), \mathbf{s})$ (see Notation 7.7) is an admissible path from v_0 to v. In this section, we give a characterization of such admissible paths.

Definition 8.4. Let $v_0 \in V_0$ and $v \in V$. For a sequence $\mathbf{s}$ of simple reflections in W, let

$$\mathcal{P}(v_0, v, \mathbf{s}) = \{(\mathbf{v}', \mathbf{s}') \in \mathcal{P}(v_0, v) : \ \mathbf{s}' \text{ is a subsequence of } \mathbf{s}\}.$$

A path $(\mathbf{v}, \mathbf{s}) \in \mathcal{P}(v_0, v)$ is said to be *minimal* if $(\mathbf{v}, \mathbf{s})$ is the only member of $\mathcal{P}(v_0, v, \mathbf{s})$. The set of all minimal admissible paths from v_0 to v will be denoted by $\mathcal{P}_{\min}(v_0, v)$.

Lemma 8.5. *Let* $v_0 \in V_0$ *and* $v \in V$.

1) If $y \in Y_{v_0}(v)$ *and* $\mathbf{s} \in R(y^{-1})$, *then* $(\mathbf{u}_{v_0,v}(\mathbf{s}), \mathbf{s}) \in \mathcal{P}_{\min}(v_0, v)$;

2) If $(\mathbf{v}, \mathbf{s}) \in \mathcal{P}_{\min}(v_0, v)$ *and* $y = y(\mathbf{v}, \mathbf{s})$, *then* $y \in Y_{v_0}(v)$, $\mathbf{s} \in R(y^{-1})$, *and* $(\mathbf{v}, \mathbf{s}) = (\mathbf{u}_{v_0,v}(\mathbf{s}), \mathbf{s})$.

Proof. 1) Let $y \in Y_{v_0}(v)$ and $\mathbf{s} \in R(y^{-1})$. Write $\mathbf{s} = (s_1, \ldots, s_k)$, and let $\mathbf{s}' = (s_{i_1}, \ldots, s_{i_p})$ be a subsequence of $\mathbf{s}$. Suppose that $(\mathbf{v}', \mathbf{s}') \in \mathcal{P}(v_0, v)$. By Lemma 8.3, $v \leq m(s_{i_p} * \cdots * s_{i_1}) \cdot v_0$. Since $s_{i_p} * \cdots * s_{i_1} \leq y$, one must have $s_{i_p} * \cdots * s_{i_1} = y$, which is possible only when $\mathbf{s}' = \mathbf{s}$. It follows that $\mathbf{v}'$ is a subexpression of the reduced decomposition $(\mathbf{v}_{v_0}(\mathbf{s}), \mathbf{s})$ of $m(y) \cdot v_0$. By Proposition 7.6, $\mathbf{v}' = \mathbf{u}_{v_0,v}(\mathbf{s})$. This proves that $(\mathbf{u}_{v_0,v}(\mathbf{s}), \mathbf{s}) \in \mathcal{P}_{\min}(v_0, v)$.

2) Let $(\mathbf{v}, \mathbf{s}) \in \mathcal{P}_{\min}(v_0, v)$ and let $y = y(\mathbf{v}, \mathbf{s})$. Let $\mathbf{s} = (s_1, \ldots, s_k)$. By Lemma 8.3, $v \leq m(s_k * \cdots * s_1) \cdot v_0$. Suppose that $s_k * \cdots * s_1 \neq s_k \cdots s_1$. Then there exists a proper subsequence $\mathbf{s}' = (s_{i_1}, \ldots, s_{i_p})$ of $\mathbf{s}$ such that $\mathbf{s}'$ is a reduced word of $y' = s_k * \cdots * s_1 = s_{i_p} \cdots s_{i_1}$. By passing to a subsequence of $\mathbf{s}'$ if necessary, we can assume that $y' \in Y_{v_0}(v)$. Then $(\mathbf{u}_{v_0,v}(\mathbf{s}'), \mathbf{s}')$ is an admissible path from v_0 to v that is different from $(\mathbf{v}, \mathbf{s})$, contradicting the assumption on $(\mathbf{v}, \mathbf{s})$. Thus $y = s_k * \cdots * s_1 = s_k \cdots s_1$, so $\mathbf{s}$ is a reduced word of y^{-1}. The same arguments show that $y \in Y_{v_0}(v)$. By Proposition 7.6, $(\mathbf{v}, \mathbf{s}) = (\mathbf{u}_{v_0,v}(\mathbf{s}), \mathbf{s})$.

Q.E.D.

For $v_0 \in V_0$ and $v \in V$, let

$$\mathcal{Y}_{v_0}(v) = \{(y, \mathbf{s}) : y \in Y_{v_0}(v), \mathbf{s} \in R(y^{-1})\}.$$

The following Proposition 8.6 follows immediately from Lemma 8.5.

Proposition 8.6. *For any $v_0 \in V_0$ and $v \in V$, the map*

$$\Omega : \quad \mathcal{P}_{\min}(v_0, v) \longrightarrow \mathcal{Y}_{v_0}(v) : \quad (\mathbf{v}, \mathbf{s}) \longmapsto (y(\mathbf{v}, \mathbf{s}), \mathbf{s}) \tag{8.1}$$

is bijective, with inverse given by

$$\Omega^{-1} : \quad \mathcal{Y}_{v_0}(v) \longrightarrow \mathcal{P}_{\min}(v_0, v) : \quad (y, \mathbf{s}) \longmapsto (\mathbf{u}_{v_0,v}(\mathbf{s}), \mathbf{s}). \tag{8.2}$$

Definition 8.7. For $v_0 \in V_0, v \in V$, and $(y, \mathbf{s}) \in \mathcal{Y}_{v_0}(v)$, we will call $(\mathbf{u}_{v_0,v}(\mathbf{s}), \mathbf{s})$ the *admissible path from v_0 to v associated to* $\mathbf{s}$.

Corollary 8.8. *For any $v_0 \in V$ and $v \in V$, one has*

$$Y_{v_0}(v) = \{y(\mathbf{v}, \mathbf{s}) : (\mathbf{v}, \mathbf{s}) \in \mathcal{P}_{\min}(v_0, v)\}.$$

8.3. **Shortest admissible paths and elements in $Z_{v_0}(v)$.** Let $v_0 \in V_0$ and $v \in V$. Let $\mathcal{P}_{\text{short}}(v_0, v)$ denote the set of all shortest admissible paths from v_0 to v (see Definition 8.1). It is clear that $\mathcal{P}_{\text{short}}(v_0, v) \subset \mathcal{P}_{\min}(v_0, v)$. Let

$$\mathcal{Z}_{v_0}(v) = \{(z, \mathbf{s}) : z \in Z_{v_0}(v), \mathbf{s} \in R(z^{-1})\}.$$

Then $\mathcal{Z}_{v_0}(v) \subset \mathcal{Y}_{v_0}(v)$.

Proposition 8.9. *For any $v_0 \in V_0$ and $v \in V$, the map Ω in* (8.1) *restricts to a bijection between $\mathcal{P}_{\text{short}}(v_0, v)$ and $\mathcal{Z}_{v_0}(v)$.*

Proof. Since $l(\mathbf{v}, \mathbf{s}) = l(y(\mathbf{v}, \mathbf{s}))$ for all $(\mathbf{v}, \mathbf{s}) \in \mathcal{P}_{\min}(v_0, v)$, Proposition 8.9 follows directly from Proposition 8.6.

Q.E.D.

Corollary 8.10. *For any $v_0 \in V_0$ and $v \in V$, one has*

$$Z_{v_0}(v) = \{y(\mathbf{v}, \mathbf{s}) : (\mathbf{v}, \mathbf{s}) \in \mathcal{P}_{\text{short}}(v_0, v)\}.$$

Example 8.11. Let $v_0 \in V_0$, and let $M(W, S) \cdot v_0 = \{m(w) \cdot v_0 : w \in W\}$. Suppose that $v \in M(W, S) \cdot v_0$. Then

$$Z_{v_0}(v) = \{z \in W : m(z) \cdot v_0 = v, l(z) = l(v)\}. \tag{8.3}$$

Indeed, if $w \in W_{v_0}(v)$, then $l(v) \leq l(m(w) \cdot v_0) \leq l(w)$. Since $v \in M(W, S) \cdot v_0$, there exists $z_0 \in W$ such that $v = m(z_0) \cdot v_0$ and $l(z_0) = l(v)$. Thus $l(v) = \min\{l(w) : w \in W_{v_0}(v)\}$. Let $Z'_{v_0}(v)$ be the set of the right hand side on (8.3). Then $Z'_{v_0}(v) \subset Z_{v_0}(v)$. Conversely, if $z \in Z_{v_0}(v)$, then $l(z) = l(v)$, and it follows from $l(v) \leq l(m(z) \cdot v_0) \leq l(z)$ that $v = m(z) \cdot v_0$. Thus $Z_{v_0}(v) \subset Z'_{v_0}(v)$. Hence $Z_{v_0}(v) = Z'_{v_0}(v)$.

We will see in §9.4 that it can happen that $v \in M(W, S) \cdot v_0$ and $Z_{v_0}(v)$ is a proper subset of $Y_{v_0}(v)$.

9. Examples

9.1. **The case when $v \in V_0$.** In this subsection, we assume that K is connected. Fix $v_0, v_0' \in V_0$. We will consider the set $Y_{v_0}(v_0')$.

Recall from §4.3 that $Y_{v_0}(v_0') = \min(W'_{v_0}(v_0'))$, where

$$W'_{v_0}(v_0') = \{w \in W: \ w = p(B', B) \text{ for some } B' \in K(v_0'),\ B \in K(v_0)\}, \tag{9.1}$$

and $p: \mathcal{B} \times \mathcal{B} \to W = (\mathcal{B} \times \mathcal{B})/G$ is the natural projection. By Lemma 6.5, $W'_{v_0}(v_0') = \{w \in W^\theta: wv_0 = v_0'\}$. Recall from Definition 6.7 the element $y_{v_0,v_0'} \in W^\theta$. By 3) of Proposition 6.8, $y_{v_0,v_0'}$ is the unique minimal element in $W'_{v_0}(v_0')$. Thus $Y_{v_0}(v_0') = \{y_{v_0,v_0'}\}$ has one element.

9.2. **The case when there is a unique closed orbit.** Assume that $V_0 = \{v_0\}$ has only one element. Then $M(W,S)\cdot v_0 = V$. In this case, for every $v \in V$, one has

$$Y_{v_0}(v) = Z_{v_0}(v) = \{z \in W: \ m(z)\cdot v_0 = v,\ l(z) = l(v)\}.$$

Indeed, by [12, Corollary 9.15], the Springer map $\phi: V \to W$ is injective. For any $v \in V$, since $\phi(s_\alpha \cdot v) = \phi(v)$ for any $\alpha \in I_v^n$, one has $I_v^{n,\neq} = \emptyset$, and thus for any admissible path $(\mathbf{v}, \mathbf{s})$ from v_0 to v, 2) in Definition 8.1 does not occur. Therefore if $y \in Y_{v_0}(v)$, then $v = m(y)\cdot v_0$ and $l(y) = l(v)$, and so $y \in Z_{v_0}(v)$.

Remark 9.1. For any $v \in V$, and not assuming that V_0 has only one element, T. A. Springer has studied in [17] the set $\cup_{v_0 \in V_0}\{z \in W: m(z)\cdot v_0 = v,\ l(z) = l(v)\}$ as an invariant for v.

9.3. **The Hermitian symmetric case.** Assume that G is simple and simply connected. By [13, Definition 5.1.1], (G,θ) is said to be of Hermitian symmetric type if the center of K has positive dimension.

Assume that (G,θ) is of Hermitian symmetric type. By [13, Theorem 5.12], there exists a standard pair $(B,H) \in \mathcal{C}$ and a parabolic subgroup P of G containing B such that K is the unique Levi subgroup of P containing H. Moreover, there exists $\alpha_0 \in \Gamma$ such that P is of type $J = \Gamma \backslash \{\alpha_0\}$. Let $v_0 \in V$ such that $B \in K(v_0)$. Then every $\alpha \in J$ is compact imaginary for v_0. We now study the set $Y_{v_0}(v)$ for every $v \in V$.

Let W_J be the subgroup of W generated by simple reflections corresponding to roots in J, and let W^J be the set of minimal length representatives for W/W_J in W. Then W^J parametrizes the set of (B,P)-double cosets in G via

$$W^J \ni d \longmapsto BdP := B\eta_{B,H}^{-1}(d)P \subset G,$$

where $\eta_{B,H}: W_H \to W$ is given in (3.4). Since every (B,K)-double coset in G is contained in a unique (B,P)-double coset, we have the well-defined surjective map

$$\nu: \ V \longrightarrow W^J: \quad \nu(v) = d \in W^J \quad \text{if} \quad BvK \subset BdP.$$

It is proved in [13, Theorem 5.2.5] that the map

$$\eta: \ V \longrightarrow W \times W^J: \ \ \eta(v) = (\phi(v), \ \nu(v))$$

is injective.

Proposition 9.2. *Let (G, θ) be of Hermitian symmetric type and let the notation be as above. Then for any $v \in V$, $\nu(v) \in W^J$ is the unique element in $Y_{v_0}(v)$.*

Proof. Note that $P = BK$. Let $v \in V$. By the definition of $\nu(v)$, one has

$$BvK \subset B\nu(v)P \subset \overline{B\nu(v)P} = \overline{B\nu(v)BK} = \overline{B\nu(v)B}\ K = \overline{B(m(\nu(v))\cdot v_0)K}.$$

Thus $\nu(v) \in W_{v_0}(v)$. Conversely, assume that $w \in W_{v_0}(v)$. Write $w = dx$, where $d \in W^J$ and $x \in W_J$. Since every $\alpha \in J$ is compact imaginary for v_0, $m(x)\cdot v_0 = v_0$. Thus $v \leq m(w)\cdot v_0 = m(d)\cdot v_0$, i.e.,

$$BvK \subset \overline{B(m(d)\cdot v_0)K} = \overline{BdB}\ K = \overline{BdP}.$$

Since $BvK \subset B\nu(v)P$, we have $B\nu(v)P \cap \overline{BdP} \neq \emptyset$. Thus $\nu(v) \leq d \leq w$. This shows that $\nu(v)$ is the unique minimal element in $W_{v_0}(v)$.

Q.E.D.

Thus, our Theorem 2.2 generalizes [13, Theorem 5.2.5].

9.4. **An example where $Z_{v_0}(v) \neq Y_{v_0}(v)$.** Let $G = SL(4, \mathbb{C})$ and let $\theta \in \mathrm{Aut}(G)$ be given by $\theta(g) = I_{2,2}(g^t)^{-1}I_{2,2}$, where for $g \in G$, g^t denotes the transpose of g, and $I_{2,2} = \mathrm{diag}(I_2, -I_2)$ with I_2 being the 2×2 identity matrix. Then $K = S(GL(2, \mathbb{C}) \times GL(2, \mathbb{C}))$. Using the "kgb" command in the *Atlas of Lie groups* (www.liegroups.org) for the real form $SU(2, 2)$ of G, one knows that there are 6 elements in V_0. Take $v_0 \in V_0$ to be the orbit labeled by 3 in the Atlas and let v be the orbit labeled by 19. With respect to v_0, the simple roots α_1 and α_3 are noncompact, while α_2 is compact, where we use the Bourbaki labeling of roots. Then $v = m(s_3s_2s_1)\cdot v_0 \in M(W, S)\cdot v_0$. By Example 8.11, $l(z) = 3$ for every $z \in Z_{v_0}(v)$.

On the other hand, let $y = s_2s_1s_2s_3 \in W$. Then $m(y)\cdot v_0 = v_{\max}$, where $v_{\max}$ is maximal element in V. Since $v \leq v_{\max}$, $y \in W_{v_0}(v)$. We claim that $y \in Y_{v_0}(v)$. Indeed, if $y \notin Y_{v_0}(v)$, then there exists $y' \in Y_{v_0}(v)$ with $y' < y$. Since $l(y) \geq l(v) = 3$, we must have $l(y) = 3$. Now there are exactly three subwords y' of the word $y = s_2s_1s_2s_3$ with length 3, namely, $y' = s_1s_2s_3$ or $s_2s_1s_3$, or $s_2s_1s_2$, and one checks directly that neither of these three choices gives $v \leq m(y')\cdot v_0$. Thus $y \in Y_{v_0}(v)$. Since $l(y) = 4$, $y \notin Z_{v_0}(v)$. We thus have an example in which $Z_{v_0}(v)$ is a proper subset of $Y_{v_0}(v)$.

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