POISSON GEOMETRY OF THE GROTHENDIECK RESOLUTION OF A COMPLEX SEMISIMPLE GROUP

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ABSTRACT. We study a Poisson structure π on the Grothendieck resolution X of a complex semi-simple group G and prove that the desingularization map $\mu : (X, \pi) \to (G, \pi_0)$ is Poisson, where π_0 is a Poisson structure such that intersections of conjugacy classes and opposite Bruhat cells BwB_- are Poisson subvarieties. We compute the symplectic leaves of X and show that (X, π) resolves singularities of (G, π_0).

CONTENTS

1. INTRODUCTION

A complex connected semi-simple group G has a well-known Poisson structure π_G introduced by Drinfeld, and the double Bruhat cells are Poisson subvarieties of (G, π_G) [9]. Let B and B_- be opposite Borel subgroups containing a maximal torus H and let U be the unipotent radical of B. In this paper, we consider a Poisson structure π_0 on G such that for every conjugacy class C of G and every w in the Weyl group W of G, the intersection $C \cap BwB_-$ is an H-orbit of symplectic leaves. Further, we construct a Poisson structure π on the Grothendieck resolution $X = G \times_B B$ of G with the following properties:

(1) The action map $(G, \pi_G) \times (X, \pi) \to (X, \pi)$ is Poisson;

(2) The morphism $\mu : (X, \pi) \to (G, \pi_0)$ given by $[g, b] \mapsto gbg^{-1}, g \in G, b \in B$ is Poisson;

(3) For $t \in H$, let $X_t = G \times_B tU$. Then X_t is a Poisson submanifold of X. Let G be simply connected. Then $\mu_t := \mu|_{X_t}$ has image the Steinberg fiber $F_t = \mu(X_t)$, and $\mu_t : X_t \to F_t$ is a desingularization. Let $X_{t,w} = X_t \cap \mu^{-1}(BwB_-)$. Then $X_{t,w}$ is a single H-orbit of symplectic leaves of π.

(4) We partition $X_{t,w}$ into Poisson subvarieties isomorphic to smooth, rational varieties.

It is well-known that X_t resolves the singularities of F_t. We remark that $(X_{t,w}, \pi)$ also resolves the singularities of the Poisson structure of $(F_t \cap BwB_-, \pi_0)$. It is convenient to specialize to the case where w is the identity. F_t is a union of finitely many conjugacy classes and there is a unique open conjugacy class R_t. While π_0 is nondegenerate at a point $g \in R_t \cap BB_-$, π_0 becomes degenerate as we move to points $g \in (F_t - R_t) \cap BB_-$. In contrast, π is nondegenerate on $X_{t,e}$, so $(X_{t,e}, \pi) \to (F_t \cap BB_-, \pi_0)$ resolves the singularity of π_0. Analogous statements are true for any $w \in W$.

This situation should be compared to the Grothendieck resolution $G \times_B \mathfrak{b} \to \mathfrak{g}$, $(g, Y) \mapsto \mathrm{Ad}_g(Y)$, for $g \in G$ and $Y \in \mathfrak{g}$. Here $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{b}$ is the Lie algebra of B, and also let $\mathfrak{h}$ and $\mathfrak{u}$ be the Lie algebras of H and U. Then $\mathfrak{g}$ has the classical Kostant-Kirillov Poisson structure, and for $Y \in \mathfrak{h}$, let $X_Y = G \times_B (Y + \mathfrak{u})$. Then X_Y, as a twisted cotangent bundle, has a well-known symplectic structure, and $X_Y \to \mathfrak{g}$, $[g, Z] \mapsto \mathrm{Ad}_g(Z)$ is Poisson [4]. The image $F_Y = \mathrm{Ad}_G(Y + \mathfrak{u})$ is Poisson and consists of finitely many G-orbits, and the rank changes depending on the dimension of the G-orbit through a point of F_Y, while X_Y is symplectic.

In the process of establishing the above results, we compute the symplectic leaves for (G, π_0) and (X, π). We also prove that for a Steinberg fiber F, $F \cap \overline{BwB_-}$ is normal and Cohen-Macaulay, compute its dimension, and estimate its singular set.

The paper is organized as follows. In section 2, we recall facts about (G, π_G) and its Drinfeld double, and use the Drinfeld double to construct π_0. We then compute symplectic leaves of π_0. In section 3, we recall facts about the Grothendieck resolution, and define π using Poisson coisotropic reduction following ideas of Alan Weinstein. We then establish several properties of (X, π) and compute its symplectic leaves. We also prove that $X_{t,w}$ is smooth. In section 4, we show how to give coordinates on $X_{t,w}$. In later work, we will find log-canonical coordinates on $X_{t,w}$ using these results, and investigate the combinatorial consequences, which should be related to work of Fomin, Kogan, and Zelevinsky in [9] and [15]. In the appendix, we recall some results from Poisson Lie theory 5.1, and establish results about coisotropic reduction needed to construct π. We further prove algebro-geometric properties about the singularities of $F \cap \overline{BwB_-}$, where F is a Steinberg fiber in 5.3.

Our construction is heavily influenced by ideas of Ginzburg, who emphasized the symplectic nature of the Grothendieck resolution for the Lie algebra in geometry and representation theory [4]. We believe our results provide the appropriate lift of these results for the Lie algebra to the case of the group. We remark that G

with its Poisson structure π_0 may be regarded as a deformation of the Lie algebra $\mathfrak{g}$ with its Kostant-Kirillov Poisson structure, and we hope to study this deformation further. When G is of adjoint type, the Poisson structure π_0 was studied in our previous papers [7] and [8], where its extension to the wonderful compactification is also discussed.

1.1. **Notation.** If G is a Lie group with Lie algebra $\mathfrak{g}$, and if $X \in \wedge^k \mathfrak{g}$, X^L and X^R will denote respectively the left and right invariant k-vector fields on G whose values at the identity element of G are X.

If π is a bi-vector field on a manifold P, $\tilde{\pi}$ will be the bundle map

$$\tilde{\pi}: \ T^*P \longrightarrow TP: \ (\tilde{\pi}(\alpha), \ \beta) = \pi(\alpha, \beta), \quad \alpha, \beta \in \Omega^1(P). \tag{1.1}$$

By a variety, we mean a complex quasi-projective variety, and a subvariety is a locally closed subset of a variety.

2. The Poisson structure π_0 on G

Let G be a connected complex semi-simple Lie group. In this section, we will recall the definition of the two Poisson structures π_G and π_0 on G. The Poisson structure π_G on G is *multiplicative* [17] and (G, π_G) is the standard Poisson Lie group structure on G [16]. The Poisson structure π_0 has the property that the conjugation action

$$(G, \pi_G) \times (G, \pi_0) \longrightarrow (G, \pi_0): \ (g, h) \longmapsto ghg^{-1}, \quad g, h \in G$$

is Poisson. The construction of π_0 is a special case of the general theory of Poisson Lie groups, which is reviewed in §5.1 in the Appendix.

2.1. **The Poisson Lie group (G, π_G) and its dual group (G^*, π_{G^*}).** Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ be the direct product Lie algebra. Let $\langle \, , \, \rangle$ be the symmetric non-degenerate bi-linear form on $\mathfrak{d}$ given by

$$\langle x_1 + y_1, \ x_2 + y_2 \rangle = \ll x_1, x_2 \gg - \ll y_1, y_2 \gg, \quad x_1, x_2, y_1, y_2 \in \mathfrak{g},$$

where $\ll \, , \, \gg$ is a *fixed* non-zero scalar multiple of the Killing form of $\mathfrak{g}$. Then clearly $\langle \, , \, \rangle$ is ad-invariant.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a choice Φ^+ of positive roots in the set Φ of roots for $(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}^\alpha$ be the corresponding root decomposition, and let

$$\mathfrak{n} = \sum_{\alpha \in \Phi^+} \mathfrak{g}^\alpha, \quad \mathfrak{n}_- = \sum_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha}.$$

The so-called *standard Manin triple* associated to $\mathfrak{g}$ (see [16]) is the quadruaple $(\mathfrak{d}, \mathfrak{g}_\triangle, \mathfrak{g}_{\rm st}^*, \langle \, , \, \rangle)$, where $\mathfrak{g}_\triangle = \{(x, x) : x \in \mathfrak{g}\}$ is the diagonal of $\mathfrak{d}$, and

$$\mathfrak{g}_{\rm st}^* = \mathfrak{h}_{-\triangle} + (\mathfrak{n} \oplus \mathfrak{n}_-) = \{(x + y, -y + x_-) : \ x \in \mathfrak{n}, x_- \in \mathfrak{n}_-, y \in \mathfrak{h}\}. \tag{2.1}$$

In particular, both $\mathfrak{g}_\triangle$ and $\mathfrak{g}_{\rm st}^*$ are maximal isotropic with respect to $\langle \, , \, \rangle$, and $\langle \, , \, \rangle$, which gives rise to a non-degenerate pairing between $\mathfrak{g}_\triangle$ and $\mathfrak{g}_{\rm st}^*$.

Let $G_\triangle = \{(g, g) : g \in G\} \subset G \times G$, and G^* be the connected subgroup of $G \times G$ with Lie algebra $\mathfrak{g}_{\rm st}^*$. Then the splitting $\mathfrak{d} = \mathfrak{g}_\triangle + \mathfrak{g}_{\rm st}^*$ gives rise to multiplicative Poisson structures π_G on $G \cong G_\triangle$ and π_{G^*} on G^* making them into a pair of dual

Poisson Lie groups [16] (see also the Appendix). If U, U_-, and H are the connected subgroups of G with Lie algebras $\mathfrak{n}, \mathfrak{n}_-$, and $\mathfrak{h}$ respectively, then

$$G^* = \{(nh, h^{-1}n_-) : \; n \in U, n_- \in U_-, h \in H\} \subset G \times G. \tag{2.2}$$

We will refer to π_G as the *standard multiplicative Poisson structure* on G.

Example 2.1. For

$$G = SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \; a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\},$$

if $\ll x, y \gg = \frac{1}{2\lambda} \operatorname{tr}(xy)$ for $x, y \in \mathfrak{sl}(2, \mathbb{C})$, where $\lambda \in \mathbb{C}, \lambda \neq 0$, then the multiplicative Poisson structure π_G on $SL(2, \mathbb{C})$ is

$$\begin{aligned} &\{b, a\}_\lambda = \lambda ab, \quad \{c, a\}_\lambda = \lambda ac, \; \{d, b\}_\lambda = \lambda bd, \quad \{d, c\}_\lambda = \lambda cd, \\ &\{d, a\}_\lambda = 2\lambda bc, \quad \{b, c\}_\lambda = 0. \end{aligned}$$

We will denote this Poisson structure by π^λ.

Notation 2.2. Throughout the paper, for each $\alpha \in \Phi^+$, we will fix a root vector $E_\alpha \in \mathfrak{g}^\alpha$ such that $\ll E_\alpha, E_{-\alpha} \gg = 1$. For $\alpha \in \Phi^+$, let $H_\alpha \in \mathfrak{h}$ be such that $\ll H_\alpha, x \gg = \alpha(x)$ for all $x \in \mathfrak{h}$, and let $\ll \alpha, \alpha \gg = \ll H_\alpha, H_\alpha \gg$. Set

$$h_\alpha = \frac{2}{\ll \alpha, \alpha \gg} H_\alpha, \quad e_\alpha = \sqrt{\frac{2}{\ll \alpha, \alpha \gg}} E_\alpha, \quad e_{-\alpha} = \sqrt{\frac{2}{\ll \alpha, \alpha \gg}} E_{-\alpha}.$$

Then the linear map $\phi_\alpha : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ given by

$$\phi_\alpha : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto h_\alpha, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e_\alpha, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto e_{-\alpha}$$

is a Lie algebra homomorphism. The corresponding Lie group homomorphism from $SL(2, \mathbb{C})$ to G will also be denoted by ϕ_α.

The following facts on π_G well-known [16].

Proposition 2.3. *1) Let* $\Lambda_0 = \frac{1}{2} \sum_{\alpha \in \Phi^+} E_\alpha \wedge E_{-\alpha} \in \wedge^2 \mathfrak{g}$. *Then (see notation in §1.1),*

$$\pi_G = \Lambda_0^L - \Lambda_0^R;$$

2) For each simple root α, *the map*

$$\phi_\alpha : \quad (SL(2, \mathbb{C}), \quad \pi^{\lambda_\alpha}) \longrightarrow (G, \pi_G)$$

is Poisson, where $\lambda_\alpha = \frac{\ll \alpha, \alpha \gg}{4}$, *and* π^{λ_α} *is the Poisson structure on* $SL(2, \mathbb{C})$ *defined in Example 2.1.*

Since π_G vanishes at elements in H, π_G is invariant under the left translation by elements in H. By an H-orbit of symplectic leaves of π_G, we mean a set of the form $\cup_{h \in H} h\Sigma$, where Σ is a symplectic leaf of π_G. For $u, v \in W$, let $G^{u,v} \subset G$ be the double Bruhat cell defined by

$$G^{u,v} = BuB \cap B_- v B_-. \tag{2.3}$$

By [9], Theorem 1.1,

$$\dim(G^{u,v}) = l(u) + l(v) + \dim(H). \tag{2.4}$$

Lemma 2.4. *(see [11, 12, 22, 15]) The* H*-orbits of symplectic leaves of* π_G *in* G *are precisely the double Bruhat cells* $G^{u,v}$ *for* $u, v \in W$.

2.2. **Definition of the Poisson structure π_0 on G.** Let $n = \dim \mathfrak{g}$. Let $\{x_j\}_{j=1}^n$ be any basis of $\mathfrak{g}_\Delta$ and let $\{\xi_j\}_{j=1}^n$ be the basis of $\mathfrak{g}_{\rm st}^*$ such that $\langle x_j, \xi_k\rangle = \delta_{jk}$ for $1 \le j, k \le n$. Define

$$\Lambda = \frac{1}{2}\sum_{j=1}^n \xi_j \wedge x_j \in \wedge^2 \mathfrak{d}.$$

Let $D = G \times G$. By §5.1, the bi-vector field π_D^+ on D given by

$$\pi_D^+ = \Lambda^R + \Lambda^L \tag{2.5}$$

is Poisson (see notation in §1.1). Let $\eta : D \to D/G_\Delta$ be given by $d \mapsto dG_\Delta$. By Proposition 5.2 in the Appendix,

$$\pi_0 := \eta(\pi_D^+)$$

is a Poisson structure on D/G_Δ.

Note that $\eta(\Lambda^L) = 0$, so

$$\pi_0 = \eta(\Lambda^R).$$

Definition 2.5. Identify D/G_Δ with G via the map

$$(G \times G)/G_\Delta \longrightarrow G : \quad (g_1, g_2)G_\Delta \longmapsto g_1 g_2^{-1}.$$

We will regard π_0 as a Poisson structure on G.

Let $\{y_i\}_{i=1}^r$ be a basis of $\mathfrak{h}$ such that $2 \ll y_i, y_j \gg = \delta_{ij}$ for $1 \le i, j \le r = \dim \mathfrak{h}$. Then for the bases of $\mathfrak{g}_\Delta$ and $\mathfrak{g}_{\rm st}^*$, we can take

$$\{x_i\} = \{(y_1, y_1),\ (y_2, y_2), \dots, (y_r, y_r),\ (E_\alpha, E_\alpha),\ (E_{-\alpha}, E_{-\alpha}) : \alpha \in \Phi^+\} \tag{2.6}$$

$$\{\xi_i\} = \{(y_1, -y_1),\ (y_2, -y_2), \dots, (y_r, -y_r),\ (0, -E_{-\alpha}),\ (E_\alpha, 0) : \alpha \in \Phi^+\}. \tag{2.7}$$

The element $\Lambda \in \wedge^2 \mathfrak{d}$ is then given by

$$\begin{aligned} \Lambda = {} & \frac{1}{2}\sum_{i=1}^r (y_i, -y_i) \wedge (y_i, y_i) \\ & + \frac{1}{2}\sum_{\alpha\in\Phi^+} \left((E_\alpha, 0) \wedge (E_{-\alpha}, E_{-\alpha}) + (0, -E_{-\alpha}) \wedge (E_\alpha, E_\alpha)\right) \end{aligned} \tag{2.8}$$

For $x \in \mathfrak{g}$, $\eta((x, 0)^R) = x^R$ and $\eta((0, x)^R) = -x^L$. It follows that

$$\pi_0 = \sum_{i=1}^r y_i^L \wedge y_i^R + \frac{1}{2}\sum_{\alpha\in\Phi^+} (E_\alpha^R \wedge E_{-\alpha}^R + E_\alpha^L \wedge E_{-\alpha}^L) + \sum_{\alpha\in\Phi^+} E_{-\alpha}^L \wedge E_\alpha^R. \tag{2.9}$$

The following Proposition 2.6 is a direct consequence of Proposition 5.2 in the Appendix.

Proposition 2.6. *The following group actions are Poisson :*

$$(G, \pi_G) \times (G, \pi_0) \longrightarrow (G, \pi_0) : \quad (g_1, g) \longmapsto g_1 g g_1^{-1} \tag{2.10}$$

$$(G^*, -\pi_{G^*}) \times (G, \pi_0) \longrightarrow (G, \pi_0) : \quad ((b, b_-), g) = bgb_-^{-1}; \tag{2.11}$$

Lemma 2.7. *The Poisson structure π_0 on G is invariant under the conjugation action by every element in H.*

Proof. This follows from (2.10) and the fact that the Poisson structure π_G vanishes at every point in H. □

Example 2.8. Let $G = SL(2,\mathbb{C})$. Using $\ll x, y \gg = \mathrm{tr}(xy)$ for $x, y \in \mathfrak{sl}(2,\mathbb{C})$, we can compute directly from (2.9) to get

$$\{a,b\} = bd, \quad \{a,c\} = -cd, \quad \{a,d\} = 0,$$
$$\{b,c\} = ad - d^2, \quad \{b,d\} = bd, \quad \{c,d\} = -cd.$$

It is easy to see that $a + d$ is a Casimir function.

2.3. **Symplectic leaves of π_0 in G.** The following Proposition 2.9 is a direct consequence of Proposition 5.2 in the Appendix. It is also proved in [1] by Alexseev and Malkin.

Proposition 2.9. *The symplectic leaves of π_0 in G are precisely the connected components of intersections of conjugacy classes in G and the G^*-orbits in G, where G^* acts on G by* (2.11).

The G^*-orbits in G for the action in (2.11) can be easily described using the Bruhat decomposition of G. Let W be the Weyl group of (G, H). For each $w \in W$, let

$$U^w = U \cap wUw^{-1} \subset U \quad \text{and} \quad H_w = \{hw(h) : h \in H\} \subset H. \tag{2.12}$$

For each $w \in W$, we will also fix a representative $\dot{w}$ in the normalizer $N_G(H)$ of H in H.

Lemma 2.10. *1) Every G^*-orbit in G for the action in* (2.11) *is of the form $G^* \cdot (h\dot{w})$ for a unique $w \in W$ and for some $h \in H$. Moreover, for any $w \in W$ and $h_1, h_2 \in H$, $G^* \cdot (h_1\dot{w}) = G^* \cdot (h_2\dot{w})$ if and only if $h_1 h_2^{-1} \in H_w$;*

2) for any $w \in W$ and $h \in H$, the map

$$U^w \times H_w \times U_- \longrightarrow G^* \cdot (h\dot{w}) : \quad (n, h_1, n_-) \longmapsto nhh_1\dot{w}n_-^{-1}$$

is a biregular isomorphism.

Proof. By the Bruhat decomposition,

$$G = \bigsqcup_{w \in W} BwB_-,$$

and for $w \in W$, BwB_- is a union of G^*-orbits. Moreover,

(2.13)
$$\phi_w : U^w \times H \times U_- \longrightarrow BwB_- : (n, h, n_-) \longmapsto nh\dot{w}n_-, \quad n \in N^w, h \in H, n_- \in U_-$$

is a biregular isomorphism. Consider a G^*-orbit $G^* \cdot x \subset BwB_-$. Since $U \times U_- \subset G^*$, $G^* \cdot x = G^* \cdot h\dot{w}$ for some $h \in H$. If $(y, y^{-1}) \in G^*$, then $(y, y^{-1}) \cdot h\dot{w} = hyw(y)\dot{w}$. The remainder of 1) follows easily, and 2) follows from the fact that ϕ_w is a biregular isomorphism. □

Remark 2.11. There exist conjugacy classes C which do not intersect every G^*-orbit in G. For example, if C is the conjugacy class of the identity element, it only intersects $G^* \cdot (h\dot{w})$ if $w = 1$. Moreover, the intersection of a conjugacy class and a G^*-orbit in G may not be connected. As an example, consider $G = SL(3,\mathbb{C})$, and let C be the conjugacy class of subregular unipotent elements, i.e.,

$$C = \{g \in SL(3,\mathbb{C}) : \ (g - I)^2 = 0, \ g \neq I\},$$

where I stands for the 3×3 identity matrix. As usual, we take B and B_- to consist of all upper and lower triangular matrices in $SL(3, \mathbb{C})$ respectively, and let H be the set of all diagonal matrices. Let w be the longest element in the Weyl group $W \cong S_3$ and take

$$\dot{w} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Fix $h = \mathrm{diag}(h_1, h_2, h_2) \in H$. It is easy to see that the G^* orbit $G^* \cdot (h\dot{w})$ in this case is given by

$$G^* \cdot (h\dot{w}) = \left\{ g = \begin{pmatrix} h_1 a & h_1 b & h_1 x \\ h_2 c & h_2 x^{-2} & 0 \\ -h_3 x & 0 & 0 \end{pmatrix} : \ a, b, c, x \in \mathbb{C}, x \neq 0 \right\}.$$

The reader can check that an element $g \in G^* \cdot (h\dot{w})$ lies in C if and only if $a = 2/h_1, b = c = 0$ and $x^2 = h_2$. Thus the intersection $C \cap (G^* \cdot (h\dot{w}))$ consists of exactly two points.

2.4. ***H*-orbits of symplectic leaves of π_0.** Since the Poisson structure π_0 is invariant under conjugation by elements in H, if Σ is a symplectic leaf of π_0, so is $h\Sigma h^{-1}$ for every $h \in H$. Let Σ be a symplectic leaf of π_0. The set

$$H \cdot \Sigma := \{h \Sigma h^{-1} : h \in H\}$$

will be referred to as an *H-orbit of symplectic leaves* of π_0.

Proposition 2.12. *1) Let C be a conjugacy class and let $w \in W$. The rank of π_0 at every point in $C \cap (BwB_-)$ is equal to*

$$\dim(C \cap (BwB_-)) - \dim(H/H_w) = \dim C - l(w) - \dim(H/H_w),$$

where $l(w)$ is the length of w.

2) The H-orbits of symplectic leaves of π_0 in G are precisely all the non-empty intersections $C \cap BwB_-$, where C is a conjugacy class in G and $w \in W$.

Proof. By Proposition 2.9, every symplectic leaf Σ in $C \cap BwB_-$ is a connected component of $C \cap G^* \cdot (h\dot{w})$, where $h \in H$. Since C intersects $G^* \cdot (h\dot{w})$ transversally,

$$\dim(\Sigma) = \dim(C) + \dim(G^* \cdot (h\dot{w})) - \dim(G) = \dim(C) - l(w) - \dim(H/H_w),$$

using Lemma 2.10 (2) to compute $\dim(G^* \cdot (h\dot{w}))$. This gives 1). To prove 2), let $C \cap G^* \cdot (h\dot{w})) = \cup \Sigma_i$ be the decomposition into connected components. We show that $H \cdot \Sigma_i = C \cap BwB_-$. Clearly, $H \cdot \Sigma_i \subset C \cap BwB_-$. By the comments preceding the proof, it follows that either $H \cdot \Sigma_i = H \cdot \Sigma_j$ or $H \cdot \Sigma_i \cap H \cdot \Sigma_j = \emptyset$.

We claim that $H \cdot (C \cap G^* \cdot (h\dot{w})) = C \cap BwB_-$. For this, consider the subgroup $H_w^- = \{hw(h^{-1}) : h \in H\}$ and note that $H = H_w H_w^-$. The image of the map

$$\alpha : H \times G^* \cdot (h\dot{w}) \to BwB_-$$

is easily seen to be $\phi_w(U^w \times H_w H_w^- \times U_-) = BwB_-$, using (2.13) and Lemma 2.10. Since C is conjugation invariant, the claim follows easily. Thus, $BwB_- = \cup H \cdot \Sigma_i$. By Proposition 4.10 of [8] with $D = G \times G$, $A = G_\Delta$, $C = B \times B_-$, and $X = G$, it follows that $C \cap BwB_-$ is smooth and connected. Thus, it suffices to prove that $H \cdot \Sigma_i$ is open in BwB_-.

Let $\beta = \phi_w^{-1} \circ \alpha : H \times \Sigma_i \to U^w \times H \times U_-$, and let $\beta_H = p_H \circ \beta$, where p_H is the projection to H. Then the differential of β_H maps the tangent space to $\{e\} \times \Sigma_i$ to $\mathfrak{h}_w$, the Lie algebra of H_w. Arguing as above, one sees the differential of β_H has image $\mathfrak{h}$. It follows that the differential of α has dimension greater than or equal to $\dim(\Sigma_i) + \dim(\mathfrak{h}/\mathfrak{h}_w) = \dim(C \cap BwB_-)$, using the first part of this proposition. It follows that $\alpha : H \times \Sigma_i \to C \cap BwB_-$ is a submersion, so $H \cdot \Sigma_i$ is open in $C \cap BwB_-$. □

By Remark 2.11, a conjugacy class in G need not intersect every Bruhat cell BwB_-. Recall [13] that a conjugacy class C in G is called regular if $\dim C = \dim G - \dim H$ [13].

Proposition 2.13. *If C is a regular conjugacy class in G, then for every $w \in W$, $C \cap (BwB_-) \neq \emptyset$.*

Proof. By Proposition 5.1 of [6], $C \cap BwB$ is nonempty. It follows easily that $C \cap Bw$ is nonempty, which implies the result. □

2.5. **The intersections of Bruhat cells and Steinberg fibres.** It is convenient to replace conjugacy classes by the associated Steinberg fibers [13]. In this subsection, we assume G is simply connected.

Let $r = \dim H$, let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be the set of simple roots, and consider the corresponding *fundamental weights* $\omega_1, \omega_2, \ldots, \omega_r$, i.e., $\omega_j \in \mathfrak{h}^*$ for each $1 \leq j \leq r$ and $\omega_j(h_{\alpha_k}) = \delta_{jk}$ for $1 \leq j, k \leq r$. Denote by χ_j the character of the irreducible representation with ω_j as the highest weight. Then the *Steinberg map* is the map [13]

$$\chi : G \longrightarrow \mathbb{C}^r : \chi(g) = (\chi_1(g), \chi_2(g), \ldots, \chi_r(g)). \tag{2.14}$$

For $z \in \mathbb{C}^r$, let

$$F_z := \chi^{-1}(z) \subset G.$$

F_z is called a *Steinberg fiber* in G. Clearly each F_z is a closed subvariety of G. The following results about Steinberg fibers are proved in §4.14 of [13].

Proposition 2.14. *For every $z \in \mathbb{C}^r$,*

1) F_z is a finite union of conjugacy classes in G;

2) F_z contains a unique regular conjugacy class R_z which open and dense in F_z;

3) F_z is irreducible and $F_z \backslash R_z$ is of codimension at least 2 in F_z.

Lemma 2.15. *For any $z \in \mathbb{C}^r$, $h \in H$, and $w \in W$, $F_z \cap (G^* \cdot (h\dot{w}))$ is nonempty. In particular, $F_z \cap BwB_-$ is nonempty.*

Proof. By Proposition 2.13, $R_z \cap BwB_- \neq \emptyset$. The second claim follows. By Proposition 2.12 2), $H \cdot (R_z \cap G^* \cdot (h\dot{w})) = R_z \cap BwB_-$, which implies the first claim. □

Corollary 2.16. *For any $z \in \mathbb{C}^r$ and $w \in W$,*

1) $F_z \cap BwB_-$ is an irreducible (possibily singular) Poisson subvariety of (G, π_0) with dimension $\dim G - \dim H - l(w)$;

2) $F_z \cap BwB_-$ is a finite union of H-orbits of symplectic leaves.

Proof. Since both F_z and BwB_- are Poisson subvarieties of G with respect to π_0, their intersection is a Poisson subvariety. The rest of 1) follows from Proposition 5.8 of the Appendix. Since F_z is a finite union of conjugacy classes, 2) follows from Proposition 2.12. □

Proposition 2.17. *(see Theorem 5.12, Proposition 5.13, and Theorem 5.14 of the Appendix) For any $z \in \mathbb{C}^r$ and $w \in W$, the set of of smooth points of $F_z \cap BwB_-$ is exactly $R_z \cap BwB_-$. Moreover, $F_z \cap BwB_-$ is normal, and is a complete intersection in BwB_-.*

Example 2.18. Consider again $G = SL(3, \mathbb{C})$ with B and B^- being the subgroup of upper and lower triangular matrices respectively. Let w_0 be the longest element in the Weyl group W. Then

$$Bw_0B_- = Bw_0 = \left\{ \begin{pmatrix} a & b & y \\ c & x & 0 \\ -\frac{1}{xy} & 0 & 0 \end{pmatrix} : a, b, c, x, y \in \mathbb{C}, x \neq 0, y \neq 0 \right\}.$$

Let $\mathcal{U}$ be the unipotent subvariety of G. Then $\mathcal{U} \cap (Bw_0)$ can be identified with the subset of $\mathbb{C}^5$ with coordinates (a, b, c, x, y) given by

$$\begin{cases} a + x = 3, \\ ax - bc + \frac{1}{x} = 3, \\ x \neq 0, y \neq 0. \end{cases}$$

By the Jacobian criterion, $\mathcal{U} \cap (Bw_0)$ is singular exactly at the subregular elements

$$\begin{pmatrix} 2 & 0 & y \\ 0 & 1 & 0 \\ -\frac{1}{y} & 0 & 0 \end{pmatrix}, \qquad y \neq 0.$$

For other w, Proposition 2.17 for $\mathcal{U} \cap BwB_-$ can be verified directly.

3. The Poisson structure π on $G \times_B B$

3.1. The Grothendieck resolution and the Springer resolution. ([24])

Let B act on $G \times B$ from the right by

$$(G \times B) \times B \longrightarrow G \times B : \quad ((g, b), b_1) \longmapsto (gb_1, b_1^{-1}bb_1). \tag{3.1}$$

Then the map

$$\mu : \quad G \times_B B \longrightarrow G : \quad [g, b] \longmapsto gbg^{-1}. \tag{3.2}$$

is called the *Grothendieck (simultaneous) resolution* of G. In §3.3, we will define a Poisson structure π on $G \times_B B$, and in §3.4, we will show that $(G \times_B B, \pi)$ can be regarded as a "desingularization" of (G, π_0). We first state some properties of the Grothendieck resolution.

Lemma 3.1. *$\mu : G \times_B B \to G$ is surjective. For $x \in G$, $\mu^{-1}(x)$ is finite if and only if x is a regular element in G.*

Recall that if V is an irreducible variety and V_s of smooth points, a *desingularization* of V is a pair (X, ξ), where X is an irreducible smooth variety and $\xi : X \to V$ is a proper morphism such that η maps $\xi^{-1}(V_s)$ isomorphically to V_s.

For each $t \in H$, since tU is invariant under the conjugation action by B, we have the smooth subvariety

$$X_t := G \times_B tU$$

of $G \times_B B$. Clearly, $G \times_B B = \bigcup_{t \in H} X_t$ is a disjoint union. Set $F_t = \mu(X_t)$. $F_t = F_{\chi(t)}$ when G is simply connected.

Proposition 3.2. *(Grothendieck, see* [24, Theorem 4.4], [25, Corollary 6.4]*) Let G be simply connected. For each $t \in H$, $(X_t, \mu|_{X_t})$ is a desingularization (called the Springer resolution) of F_t.*

For $t \in H$ and $w \in W$, let

$$F_{t,w} = F_t \cap BwB_- \subset G,$$

By Example 2.18, $F_{t,w}$ is in general a singular variety.

3.2. **A desingularization of $F_{t,w}$.**

Definition 3.3. For $t \in H$ and $w \in W$, let

$$X_{t,w} = X_t \cap \mu^{-1}(BwB_-) \subset G \times_B B.$$

Clearly $\mu(X_{t,w}) = F_{t,w}$. In particular, $X_{t,w} \neq \emptyset$ for any $t \in H$ and $w \in W$ by Lemma 2.15.

Theorem 3.4. *For any $t \in H$ and $w \in W$, $X_{t,w}$ is smooth. If G is simply connected, $(X_{t,w}, \mu|_{X_{t,w}})$ is a desingularization of $F_{t,w}$. In particular,*

$$\dim X_{t,w} = \dim G - \dim H - l(w).$$

Proof. It suffices to prove $X_{t,w}$ is smooth, which follows from the following Lemma with $X = X_t$, $Z = G$, $Y = BwB_-$, and $f = \mu$. At $y = \mu(x) \in Y$, let C_y be the conjugacy class of y. Then $T_y(C_y) \subset \mu_*(T_x(X_t))$, so

$$T_y(G) = T_y(C_y) + T_y(ByB_-) \subset \mu_*(T_x(X_t)) + T_y(ByB_-) \subset T_y(G),$$

and the hypothesis of the Lemma is satisfied. □

Lemma 3.5. *Let $f : X \to Z$ be a morphism of smooth algebraic varieties, and let $Y \subset Z$ be a smooth subvariety. Suppose for all $x \in f^{-1}(Y)$, $f_*(T_x(X)) + T_{f(x)}(Y) = T_{f(x)}(Z)$. Then $f^{-1}(Y)$ is smooth.*

3.3. **Definition of the Poisson structure π on $G \times_B B$.** Recall that π_D^+ is the Poisson structure on $G \times G$ given in (2.5). Let ϕ be the projection

$$\phi : \quad G \times G \longrightarrow (G \times G)/B_\Delta.$$

Lemma 3.6. *$\pi := \phi(\pi_D^+)$ is a well-defined Poisson structure on $(G \times G)/B_\Delta$. Moreover, equip $G_\Delta \cong G$ with the multiplicative Poisson structure π_G. Then the action of $(G_\Delta,\ \pi_G)$ on $((G \times G)/B_\Delta,\ \pi)$ by left multiplication is a Poisson action.*

Proof. It is well-known [16, 10] that B_Δ is a Poisson subgroup of $(G_\Delta,\ \pi_G)$. By Lemma 5.1 in the Appendix, the action of (B_Δ, π_G) on $(G \times G, \pi_D^+)$ by right multiplication is a Poisson action. Thus (see [17] or Proposition 5.6) $\pi := \phi(\pi_D^+)$ is a well-defined Poisson structure on $(G \times G)/B_\Delta$. Again by Lemma 5.1 in the Appendix, the action of $(G_\Delta,\ \pi_G)$ on $((G \times G)/B_\Delta,\ \pi))$ by left multiplication is a Poisson action. □

We will define a Poisson structure π on $G \times_B B$ as a coisotropic reduction of π_D^+. For this, we embed $G \times_B B$ into $(G \times G)/B_\Delta$ via the following construction. Consider the biregular isomorphism

$$\psi : \quad G \times G \longrightarrow G \times G : \quad (g,\ x) \longmapsto (gx,\ g). \tag{3.3}$$

Then for $b \in B$,

$$\psi(gb, b^{-1}xb) = \psi(g, x) \cdot (b, b).$$

For $t \in H$, let $Q_t = \psi(G \times tU)$ and let $Q = \psi(G \times B)$. Then

$$Q_t = \{(gtn,\, g) : \ g \in G, n \in U\}. \tag{3.4}$$

$$Q = \cup_{t \in H} Q_t = \{(gb,\, g) : g \in G, b \in B\} \subset G \times G. \tag{3.5}$$

Thus, Q is stable under right multiplication by elements of B_Δ, and ψ descends to a biregular isomorphism, also denoted ψ, $\psi : G \times_B B \to Q/B_\Delta$. In particular, $\psi(X_t) = Q_t/B_\Delta$.

Consider the morphism $\tilde{\tau} : G \times G \to G$, $\tilde{\tau}(g, x) = gx^{-1}$ and define $\tau : (G \times G)/B_\Delta \to G$ by the same formula. Note that $\tilde{\tau} \circ \psi = \tilde{\mu}$, where $\tilde{\mu} : G \times G \to G$ is the morphism $(x, y) \mapsto xyx^{-1}$. Since $G \times G = \cup_{w \in W} (B \times B_-)(w, e)G_\Delta$ and $\tilde{\tau}((B \times B_-)(w, e)G_\Delta) = BwB_-$, it follows that $\tilde{\tau}^{-1}(BwB_-) = (B \times B_-)(w, e)G_\Delta$. Thus,

$$\psi(\mu^{-1}(BwB_-)) = (Q \cap \tau^{-1}(BwB_-))/B_\Delta = (Q \cap (B \times B_-)(w, e)G_\Delta)/B_\Delta. \tag{3.6}$$

Similarly, for all $h \in H$ and the representatives $\dot{w}$ for $w \in W$ from section 2.3,

$$\psi(\mu^{-1}(G^* \cdot (h\dot{w})) = (Q \cap G^*(hw, e)G_\Delta)/B_\Delta. \tag{3.7}$$

In particular, $Q \cap ((B \times B_-)(w, e)G_\Delta) \neq \emptyset$ for any $w \in W$.

We will show that $Q/B_\Delta \cong G \times_B B$ is a Poisson submanifold of $(G \times G)/B_\Delta$ with respect to π. We will do this by using Proposition 5.6 in the Appendix. We first need some lemmas. Recall the definition of a coisotropic submanifold of a Poisson manifold from §5.2 in the Appendix.

Lemma 3.7. *For any $t \in H$, tU is a coisotropic submanifold of (G, π_0).*

Proof. By Remark 5.4, it suffices to show that $\pi_0(b) \in T_b(tU) \wedge T_bG$ for every $b = tn \in tU$ with $n \in U$. We will use the formula for π_0 in (2.9). It is easy to see that for each $\alpha \in \Phi^+$, $E^R_\alpha(b) \in T_b(tU)$ and $E^L_\alpha(b) \in T_b(tU)$. Moreover, for $1 \leq i \leq r$, since $(y^R_i - y^L_i)(b) \in T_b(tU)$,

$$y^L_i(b) \wedge y^R_i(b) = y^L_i(b) \wedge (y^R_i - y^L_i)(b) \in T_b(tU) \wedge T_bG.$$

Hence $\pi_0(b) \in T_b(tU) \wedge T_bG$. □

Recall that for $t \in H$, $Q_t \subset G \times G$ is given in (3.4).

Lemma 3.8. *For any $t \in H$, $Q_t \subset G \times G$ is coisotropic with respect to the Poisson structure π^+_D.*

Proof. Consider the map

$$\eta_1 : \quad G \times G \longrightarrow G : \quad (g_1, g_2) \longmapsto g_1^{-1}g_2.$$

By the definition of π_0, the map

$$\eta : \quad G \times G \longrightarrow G : \quad (g_1, g_2) \longmapsto g_1g_2^{-1}$$

is Poisson. Let $\tau : G \times G \to G \times G : d \mapsto d^{-1}$. Then it is clear from the definition of π^+_D that τ preserves π^+_D. Thus $\eta_1 = \eta \circ \tau$ is Poisson.

Let $t \in H$. By Lemma 3.7, $Ut^{-1} \subset G$ is a coisotropic submanifold of G with respect to the Poisson structure π_0. Note that $Q_t = \eta_1^{-1}(Ut^{-1})$. Since $\eta_1 : (G \times$

$G, \pi_D^+) \to (G, \pi_0)$ is a Poisson submersion, it follows from [26] that $\eta_1^{-1}(Ut^{-1})$ is coisotropic in D with respect to π_D^+. □

Recall from §5.2 that the characteristic distribution of π_D^+ on Q_t is by definition the image of the bundle map

$$\tilde{\pi}_D^+ : \ (TQ_t)^\perp \longrightarrow TQ_t,$$

where $(TQ_t)^\perp$ is the conormal bundle of Q_t in $G \times G$, and $\tilde{\pi}_D^+ : T^*(G\times G) \to T(G \times G)$ is defined as in (1.1).

Lemma 3.9. *For any $t \in H$, the characteristic distribution of π_D^+ on Q_t coincides with that defined by the B_Δ-action on Q_t by right multiplication.*

Proof. Fix $g \in G$ and $n \in U$ and let $d = (gtn, \, g) \in Q_t$. We will use formula (5.4) in the proof of Lemma 5.3 in the Appendix to compute $\tilde{\pi}_D^+((T_dQ_t)^\perp)$. First, we have

$$T_dQ_t = r_d \mathfrak{g}_\Delta + l_d(\mathfrak{n} \oplus 0) = r_d \left(\mathrm{Ad}_{(g,g)}(\mathfrak{g}_\Delta + (\mathfrak{n} \oplus 0)) \right).$$

Identify $\mathfrak{d}$ with $\mathfrak{d}^*$ via the bilinear form $\langle \, , \, \rangle$, and for $(x, y) \in \mathfrak{d}$, let $\alpha_{(x,y)}$ be the right invariant 1-form on D whose value at the identity element is (x, y). Then

$$(T_dQ_t)^\perp = \{\alpha_{(x,x)}(d) : \ \ x = \mathrm{Ad}_g y \text{ with } y \in \mathfrak{b}\}.$$

For $x = \mathrm{Ad}_g y$ with $y \in \mathfrak{b}$, by formula (5.3) in the proof of Lemma 5.3 in the Appendix,

$$\tilde{\pi}_D^+(\alpha_{(x,x)}(d)) = l_d p_1 \mathrm{Ad}_d^{-1}(x, x),$$

where $p_1 : \mathfrak{d} \to \mathfrak{g}_\Delta$ is the projection with respect to the decomposition $\mathfrak{d} = \mathfrak{g}_\Delta + \mathfrak{g}_{\mathrm{st}}^*$. Since $y \in \mathfrak{b}$, we have

$$p_1 \mathrm{Ad}_d^{-1}(x, x) = p_1(\mathrm{Ad}_{tn}^{-1} y, \ y) = (y, \ y).$$

Thus $\tilde{\pi}_D^+(\alpha_{(x,x)}(d)) = l_d(y, \ y)$, which is exactly the infinitesimal generator of the B action on Q_t by right translation in the direction of y. □

Theorem 3.10. *Q/B_Δ is a Poisson submanifold of $(G \times G)/B_\Delta$ with respect to the Poisson structure π in Lemma 3.6.*

Proof. For each $t \in H$, it follows from Proposition 5.6 that Q_t/B_Δ is a Poisson submanifold of $(G \times G)/B_\Delta$ with respect to π. Since

$$Q/B_\Delta = \bigcup_{t \in H} Q_t/B_\Delta,$$

it follows that Q/B_Δ is a Poisson submanifold of $((G \times G)/B_\Delta, \, \pi)$. □

Definition 3.11. Identify $G \times_B B$ with Q/B_Δ via the isomorphism ψ in (3.3). The Poisson structure on $G \times_B B$ corresponding to the Poisson structure π on Q/B_Δ will also be denoted by π.

We now summarize some of the properties of the Poisson structure π on $G \times_B B$. Recall the Grothendieck resolution $\mu : G \times_B B \to G$ in (3.2). Let σ be the left action of G on $G \times_B B$ given by

$$\sigma : \ \ g_1 \cdot [g, b]) = [g_1 g, b], \qquad g, g_1 \in G, b \in B. \tag{3.8}$$

For each $t \in H$, recall from §3.1 that $X_t = G \times_B tU \subset G \times_B B$. Let $p : G \times_B B \to G/B$ be the projection $[g, b] \mapsto gB$ for $g \in G, b \in B$.

Proposition 3.12. *1) The Grothendieck map $\mu : (G \times_B B, \pi) \to (G, \pi_0)$ is a Poisson map;*

2) With the Poisson structure π on $G \times_B B$, σ is a Poisson action by the Poisson Lie group (G, π_G);

3) For each $t \in H$, X_t is a Poisson submanifold of $G \times_B B$ with respect to the Poisson structure π;

Proof. 1) follows from the definition of π and the definition of π_0 in §2.2, 2) follows from Lemma 5.1 in the Appendix, and 3) follows from the definition of π and the proof of Theorem 3.10. □

We now prove one more property of π. Let $q : G \to G/B$ be the projection $g \mapsto gB$. Since B is a Poisson subgroup of (G, π_G), $q(\pi_G)$ is a well-defined Poisson structure on G/B, making $(G/B, q(\pi_G))$ a Poisson (G, π_G)-homogeneous space. In particular, $q(\pi_G)$ is invariant under the left translation by elements in H.

Lemma 3.13. [10] *The H-orbits of symplectic leaves of $q(\pi_G)$ in G/B are precisely the intersections $(Bu.B) \cap (B_-v.B)$ for $u, v \in W, v \le u$.*

Proposition 3.14. *The projection $p : (G \times_B B, \pi) \to (G/B, q(\pi_G))$ is Poisson.*

Proof. By the definition of π_D^+, the formula (2.8) for Λ, and Remark 5.4, $B \times B$ is a coisotropic submanifold of $G \times G$ with respect to π_D^+. Thus by Corollary 2.25 of [26], $(B \times B)/B_\triangle$ is coisotropic in $(G \times G)/B_\triangle$ with respect to π, and hence $(B \times B)/B_\triangle$ is coisotropic in $(Q/B_\triangle, \pi)$. Since $\psi(B \times_B B) = (B \times B)/B_\triangle$, $B \times_B B$ is coisotropic in $G \times_B B$ with respect to π. It follows that for any $x = [e, b] \in G \times_B B$ with $b \in B$, $p(\pi(x)) = 0$. It follows that both $q(\pi_G)$ and $p(\pi)$ vanish at eB. Since the action σ is Poisson, it follows that $p(\pi)$ is (G, π_G) homogeneous, and since $q(\pi_G)$ is obviously (G, π_G) homogeneous, $q(\pi_G) = p(\pi)$. □

3.4. **Symplectic leaves of π in $G \times_B B$.** Let $t \in H$. By Proposition 3.12, $X_t = B \times_B tU$ is a Poisson submanifold of π in $G \times_B B$. By Proposition 5.6, to understand the symplectic leaves of π on $G \times_B tU \cong Q_t/B_\triangle$ for $t \in H$, it suffices to understand intersections of Q_t with symplectic leaves of π_D^+ in $D = G \times G$.

Recall that $\dot{w}$ is a representative for w in the normalizer $N_G(H)$ of H in G.

Lemma 3.15. *Symplectic leaves of π_D^+ are connected components of non-empty intersections*

$$\Sigma_{w,u,h,h'} := G^*(h\dot{w}, e)G_\triangle \cap G_\triangle(e, h'\dot{u})G^*,$$

where $h, h' \in H$ and $w, u \in W$. Moreover, $\Sigma_{w,u,h,h'} = \Sigma_{w_1,u_1,h_1,h'_1}$ if and only if $w = w_1, u = u_1, hh_1^{-1} \in H_w$ and $h'(h'_1)^{-1} \in H_u$, where H_w and H_u are defined in (2.12).

Proof. By Lemma 5.3 in the Appendix, symplectic leaves of π_D^+ are the connected components of non-empty intersection of $(G^*, G_\triangle)$ and $(G_\triangle, G^*)$ double cosets in $G \times G$. By Lemma 2.10, $(G^*, G_\triangle)$ and $(G_\triangle, G^*)$ double cosets in $G \times G$ are respectively of the form $G^*(h\dot{w}, e)G_\triangle$ and $G_\triangle(e, h'\dot{u})G^*$, where $h, h' \in H$ and $w, u \in W$. The rest of Lemma 3.15 also follows from Lemma 2.10. □

Now for $t \in H$, we note that $Q_t \subset G_\triangle G^*$. Indeed, let $t_1 \in H$ be such that $t_1^2 = t$. Then for any $g \in G$ and $n \in U$, $(gtn, g) = (gt_1, gt_1)(t_1 n, t_1^{-1}) \in G_\triangle G^*$. For each

$w \in W$ and $h \in H$, let

$$\Sigma_{h\dot{w}} = G^*(h\dot{w}, e)G_\triangle \cap G_\triangle G^* \subset G \times G. \tag{3.9}$$

Then Q_t can only have non-empty intersections with symplectic leaves of π_D^+ lying inside the $\Sigma_{h\dot{w}}$'s. Recall the subgroups U^w and H_w of G from (2.12).

Lemma 3.16. *For $h, t \in H$, and $w \in W$, $Q_t \cap \Sigma_{h\dot{w}} \neq \emptyset$, and the intersection is transversal. Moreover,*

$$\dim(Q_t \cap \Sigma_{h\dot{w}}) = \dim G + \dim U^w - \dim(H/H_w).$$

Proof. Since $Q_t = \tilde{\tau}^{-1}(F_{\chi(t)})$ and $G^*(h\dot{w}, e)G_\triangle = \tilde{\tau}^{-1}(G^* \cdot (h\dot{w}))$, $Q_t \cap G^*(h\dot{w}, e)G_\triangle$ is nonempty by Lemma 2.15. Since $Q_t \subset G_\triangle G^*$, $Q_t \cap \Sigma_{h\dot{w}}$ is nonempty. To check the intersection is transversal, note that for any $d \in Q_t \cap \Sigma_{h\dot{w}}$,

$$T_d Q_t + T_d \Sigma_{h\dot{w}} \supset r_d \mathfrak{g}_\triangle + r_d \mathfrak{g}^* = r_d \mathfrak{d} = T_d D.$$

Since $\tilde{\tau}$ is a $G_\triangle$-fiber bundle and $G_\triangle G^*$ is open in D, $\dim(\Sigma_{h\dot{w}}) = \dim(G) + \dim(G^* \cdot (h\dot{w}))$. Thus, by Lemma 2.10,

$$\dim(Q_t \cap \Sigma_{h\dot{w}}) = \dim Q_t + \dim \Sigma_{h\dot{w}} - \dim D = \dim G + \dim U^w - \dim(H/H_w).$$

□

Recall that for $h \in H$ and $w \in W$, $G^* \cdot (h\dot{w})$ denotes the G^*-orbit in G through $h\dot{w}$, where the group G^* acts on G by (2.11).

Corollary 3.17. *The symplectic leaves of π in $G \times_B B$ are precisely the connected components of the intersections $X_t \cap \mu^{-1}(G^* \cdot (h\dot{w}))$, where $t, h \in H, w \in W$. Moreover,*

$$\dim(X_t \cap \mu^{-1}(G^* \cdot (h\dot{w}))) = 2l(w_0) - l(w) - \dim(H/H_w).$$

Proof. By Proposition 5.6, the fact that $Q_t \subset G_\triangle G^*$ and Lemma 3.15, symplectic leaves in $Q_t/B_\triangle$ are connected components of $(\Sigma_{h\dot{w}} \cap Q_t)/B_\triangle$. Since $\psi(\mu^{-1}(G^* \cdot (h\dot{w}))) = \Sigma_{h\dot{w}}/B_\triangle$ and $\psi(X_t) = Q_t/B_\triangle$, the description of symplectic leaves follows. The dimension formula is an easy consequence. □

Remark 3.18. We recall that a Poisson variety (Z, π) is called regular if π has constant rank. Thus, Corollary 3.17 implies that $X_t \cap \mu^{-1}(G^* \cdot (h\dot{w}))$ is regular. When G is simply connected, $\mu : X_t \cap \mu^{-1}(G^* \cdot (h\dot{w})) \to F_t \cap G^* \cdot (h\dot{w})$ may be regarded as a resolution of the singular Poisson structure on $F_t \cap G^* \cdot (h\dot{w})$.

3.5. **H-orbits of symplectic leaves of π in $G \times_B B$.** Recall that the Poisson structure π_G on G vanishes at points in H. Thus the action of H on $G \times_B B$ by σ in (3.8) preserves the Poisson structure π. Recall also that a smooth Poisson variety (X, π) is called regular if π has constant rank. The following Corollary 3.19 is a direct consequence of Corollary 3.17.

Corollary 3.19. *The H-orbits of symplectic leaves of π in $G \times_B B$ are precisely the smooth irreducible subvarieties $X_{t,w}$, where $t \in H$ and $w \in W$. The dimension of the symplectic leaves in $X_{t,w}$ is $2l(w_0) - l(w) - \dim(H/H_w)$. In particular, $(X_{t,w}, \pi)$ is a regular Poisson variety.*

4. Geometry of $X_{t,w}$

In this section we will construct for every $t \in H$ and $w \in W$ an open embedding from ${\mathbb{C}^*}^{2l(w_0)-l(w)}$ to $X_{t,w}$. In a later paper, we will use these results to construct a set of $2l(w_0) - l(w)$ log-canonical rational functions on $X_{t,w}$ with respect to the Poisson structure π. In [15], this problem is resolved for double Bruhat cells $G^{u,v}$.

4.1. **Relation between X_{t,w_0} and $X_{t,w}$.** Let w_0 be the longest element in W. We relate $X_{t,w}$ with X_{t,w_0} for any $w \in W$. Recall that $\sigma : G \times (G \times_B B) \to G \times_B B$ is the action of G on $G \times_B B$ given by

$$\sigma(g_1, [g, b]) = [g_1 g, b], \qquad g_1, g \in G, b \in B.$$

For $u, v \in W$, recall that the double Bruhat cell $G^{u,v} = BuB \cap B_-vB_- \subset G$ is an H-orbit of symplectic leaves of π_G, where H acts on G by left translation (see Lemma 2.4).

Lemma 4.1. *For any $w \in W$,*

$$\sigma\left(G^{1,w^{-1}w_0} \times X_{t,w_0}\right) \subset X_{t,w}.$$

Proof. Let $[(g, tn)] \in X_{t,w_0}$, so $g \in G$, $n \in U$, and $gtng^{-1} \in Bw_0$. If $g_1 \in G^{1,w^{-1}w_0}$, $g_1^{-1} \in B_-w_0^{-1}wB_-$, so since $Bw_0B_- = Bw_0$,

$$\mu([g_1g, b]) = g_1gtng^{-1}g_1^{-1} \in Bw_0B_-w_0^{-1}wB_- = BwB_-.$$

□

Since $G^{1,w^{-1}w_0}$ and X_{t,w_0} are Poisson submanifolds of (G, π_G) and (X_t, π) (see Lemma 2.4 and Corollary 3.19), Proposition 3.12 implies the following result.

Corollary 4.2. *For $w \in W$, equip $G^{1,w^{-1}w_0}$ with the Poisson structure π_G and $X_{t,w}$ the Poisson structure π. Then*

$$\sigma : \ (G^{1,w^{-1}w_0}, \pi_G) \times (X_{t,w_0}, \pi) \longrightarrow (X_{t,w}, \pi)$$

is a Poisson map.

4.2. **A decomposition of X_{t,w_0}.** We now partition X_{t,w_0} into smooth rational subvarieties. We first set up some notation.

Notation 4.3. Let $G_0 = B_-B$. For $g \in B_-B$, we write

$$g = [g]_-[g]_0[g]_+, \qquad \text{where } [g]_- \in U_-, [g]_0 \in H, [g]_+ \in U.$$

For $v \in W$, let $U_v = U \cap vU_-v^{-1}$. Then it is well-known that

$$U_v \longrightarrow Bv.B : \ m \longmapsto m\dot{v}.B$$

is a biregular isomorphism for any $v \in W$.

Fix $t \in H$. Recall that

$$X_{t,w_0} = \{[g, tn] \in G \times_B B : g \in G, n \in U, gtng^{-1} \in Bw_0\}.$$

By Theorem 3.4, $\dim X_{t,w_0} = l(w_0)$. Let again $p : G \times_B B \to G/B : [g, b] \mapsto g.B$ be the projection. For $u \in W$, set

$$X_{t,w_0}^u = X_{t,w_0} \cap p^{-1}(Bw_0u \cdot B). \tag{4.1}$$

By the Bruhat decomposition, we have the disjoint union

$$X_{t,w_0} = \bigsqcup_{u \in W} X_{t,w_0}^u. \tag{4.2}$$

The following result is well-known, and follows from [5], Corollary 1.2, and [8], Proposition 4.10.

Proposition 4.4. *For $u, v \in W$, $B_-u.B \cap Bv.B$ is nonempty if and only if $u \leq v$. If nonempty, it is smooth and connected, and its dimension is $l(v) - l(u)$. In particular, $B_-u.B \cap Bu.B = u.B$.*

Lemma 4.5. *Let $u \in W$. If $X_{t,w_0}^u \neq \emptyset$, then $u \leq w_0 u$ and*

$$p(X_{t,w_0}^u) \subset (B_-u.B) \cap (Bw_0u.B).$$

Proof. Assume that $[g, tn] \in X_{t,w_0}^u$, where $g \in G$ and $n \in U$. Since $g \in Bw_0uB$ and $gtng^{-1} \in Bw_0$, $g \in (Bw_0)^{-1}Bw_0uB = B_-uB$. Thus

$$g \in B_-uB \cap Bw_0uB.$$

By Proposition 4.4, $u \leq w_0 u$. □

For $u \in W$ such that $u \leq w_0 u$, set

$$\mathcal{R}_{u,w_0u} = (B_-u.B) \cap (Bw_0u.B) \subset G/B.$$

Consider the isomorphism $\psi_{w_0u} : U_{w_0u} \to Bw_0u.B : m \to m\dot{w}_0\dot{u}.B$. Note that

$$\psi_{w_0u}(U_{w_0u} \cap B_-uBu^{-1}w_0^{-1}) = \mathcal{R}_{u,w_0u},$$

as can be shown using the Bruhat decomposition $U_{w_0u} = \cup_{v\in W} U_{w_0u} \cap B_-vBu^{-1}w_0^{-1}$. Recall that

$$B_- \times U \cap u^{-1}Uu \to B_-uB, \; (x, y) \mapsto x\dot{u}y$$

is an isomorphism of varieites. Thus, for $m \in U_{w_0u} \cap B_-uBu^{-1}w_0^{-1}$,

$$m\dot{w}_0\dot{u} = x\dot{u}\xi^u(m), \text{for unique } x \in B_-, \xi^u(m) \in U \cap u^{-1}Uu.$$

It follows that $m\dot{w}_0 \in G_0$, and

$$\xi^u : \;\; U_{w_0u} \cap B_-uBu^{-1}w_0^{-1} \longrightarrow U \cap u^{-1}Uu : \;\; m \longmapsto \dot{u}^{-1}[m\dot{w}_0]_+\dot{u}. \tag{4.3}$$

Define

(4.4)

$$J_t^u : \mathcal{R}_{u,w_0u} \times (U \cap u^{-1}U_-u) \longrightarrow X_t : \; (m\dot{w}_0\dot{u}.B, \, m_1) \longmapsto [(m\dot{w}_0\dot{u}, \; tm_1\xi^u(m))]$$

for $m \in U_{w_0u} \cap B_-uBu^{-1}w_0^{-1}$ and $m_1 \in U \cap u^{-1}U_-u$.

Proposition 4.6. *For any $u \in W$ such that $u \leq w_0 u$, J_t^u is an embedding and the image of J_t^u is X_{t,w_0}^u. In particular, X_{t,w_0}^u is smooth and irreducible and*

$$\dim X_{t,w_0}^u = l(w_0) - l(u).$$

Proof. Since

$$U_{w_0u} \times U \longrightarrow X_t : \;\; (m, \, n) \longmapsto [m\dot{w}_0\dot{u}, \; tn]$$

is an embedding, it follows that J_t^u is an embedding. We show that $\mathrm{Im} J_t^u \subset X_{t,w_0}^u$.

Let $m \in U_{w_0 u} \cap B_- u B u^{-1} w_0^{-1}$, and write $m\dot{w}_0\dot{u} = b_-\dot{u}\xi^u(m)$ for unique $b_- \in B_-$. Then for any $m_1 \in U \cap \dot{u}^{-1}U_-\dot{u}$,

$$(m\dot{w}_0\dot{u})(tm_1\xi^u(m))(m\dot{w}_0\dot{u})^{-1} = m\dot{w}_0\dot{u}tm_1\xi^u(m)\xi^u(m)^{-1}\dot{u}^{-1}b_-^{-1} \in Bw_0B_- = Bw_0.$$

It follows easily that $\mathrm{Im} J_t^u \in X_{t,w_0}^u$.

Define an inverse to J_t^u as follows. Let $[(g,tn)] \in X_{t,w_0}^u$. By Lemma 4.5, we can assume without loss of generality that $g = m\dot{w}_0\dot{u}$ with $m \in U_{w_0u} \cap B_-uBu^{-1}w_0^{-1}$. Uniquely factor

$$n = m_1 n_1, \quad m_1 \in U \cap u^{-1}U_-u, \quad n_1 \in U \cap u^{-1}Uu.$$

Let

$$K_t^u([(m\dot{w}_0\dot{u}, tm_1n_1)]) = (m\dot{w}_0\dot{u}.B, m_1).$$

Let $[(g, tm_1n_1)] \in X_{t,w_0}^u$ with $g = m\dot{w}_0\dot{u}$ as above. Since $gtng^{-1} \in Bw_0$,

$$g \in \dot{w}_0^{-1}Bm\dot{w}_0\dot{u}tn = B_-\dot{u}tm_1n_1 = B_-\dot{u}n_1.$$

Thus, by the definition of ξ^u, $n_1 = \xi^u(m)$, and it follows that K_t^u and J_t^u are inverse isomorphisms.

By Proposition 4.4, it follows that X_{t,w_0}^u is smooth and irreducible, and that

$$\dim X_{t,w_0}^u = l(w_0u) - l(u) + l(u) = l(w_0) - l(u).$$

□

When $u = 1 \in W$, the open subset X_{t,w_0}^1 of X_{t,w_0} is especially simple. Indeed, let

$$\xi := \xi^1 : \ U \cap B_-w_0B_- \longrightarrow U : \ m \longmapsto [m\dot{w}_0]_+. \tag{4.5}$$

Then using the biregular isomorphism $U \cap B_-w_0B_- \to \mathcal{R}_{1,w_0} : m \mapsto m\dot{w}_0.B$, the isomorphism $J_t := J_t^1$ in (4.4) simplifies to

$$J_t : \ \mathcal{R}_{1,w_0} \longrightarrow X_{t,w_0}^1 : \ m\dot{w}_0.B \longmapsto [m\dot{w}_0, \ t\xi(m)], \qquad m \in U \cap B_-w_0B_- \tag{4.6}$$

whose inverse is the restriction to $X_{t,w}^1$ of the projection $p : G \times_B B \to G/B : [g,b] \mapsto gB$. The following Lemma 4.7 can be checked directly.

Lemma 4.7. *The image of ξ in (4.5) is again $U \cap B_-w_0B_-$ and*

$$\xi : \ U \cap B_-w_0B_- \longrightarrow U \cap B_-w_0B_- : \ m \longmapsto [m\dot{w}_0]_+ \tag{4.7}$$

is biregular with inverse given by

$$\eta = \xi^{-1} : \ U \cap B_-w_0B_- \longrightarrow U \cap B_-w_0B_- : \ n \longmapsto [n\dot{w}_0^{-1}]_+. \tag{4.8}$$

Moreover, for $m \in U \cap B_-w_0B_-$, $\xi(tmt^{-1}) = w_0(t)\xi(m)w_0(t^{-1})$.

Since X_{t,w_0}^1 is open in the Poisson subvariety (X_{t,w_0}, π), X_{t,w_0}^1 is a Poisson subvariety of $(G \times_B B, \pi)$. On the other hand, by Lemma 2.4, $G^{1,w_0} = B \cap B_-w_0B_-$ is a Poisson subvariety of (G, π_G) Since π_G is invariant under right multiplication by elements in H, the quotient $G^{1,w_0}/H$ has a well-defined Poisson structure which we will still denote by π_G.

Lemma 4.8. *The map*

$$\tilde{J}_t : \ \left(G^{1,w_0}/H, \ \pi_G\right) \longrightarrow (X_{t,w_0}^1, \ \pi) : \ gH \longmapsto J_t(g\dot{w}_0.B) \tag{4.9}$$

is a Poisson isomorphism.

Proof. By Lemma 3.13, it follows that $\mathcal{R}_{1,w_0}$ is a Poisson subvariety of $(G/B, q(\pi_G))$, and using Proposition 4.4, $q(\pi_G)$ vanishes at $\dot{w}_0.B \in G/B$. Since the action of (G, π_G) on $(G/B,\ q(\pi_G))$ by left translation is Poisson, the map

$$(G^{1,w_0},\ \pi_G) \longrightarrow (\mathcal{R}_{1,w_0},\ q(\pi_G)):\ \ g \longmapsto g\dot{w}_0.B, \qquad g \in G^{1,w_0}$$

is Poisson. The projection $(X^1_{t,w_0}, \pi) \to (\mathcal{R}_{1,w_0}, q(\pi_G))$ is Poisson by Proposition 3.14, so its inverse

$$J_t:\ (\mathcal{R}_{1,w_0},\ q(\pi_G)) \longrightarrow (X^1_{t,w_0},\ \pi),$$

is a Poisson isomorphism. The result follows. □

Consider the map

(4.10)

$$\rho_t:\ (G^{1,w^{-1}w_0},\ \pi_G) \times_H (G^{1,w_0}/H,\ \pi_G) \longrightarrow (X_{t,w}, \pi) : (g_1, g_2) \longmapsto \sigma(g_1, \tilde{J}_t(g_2\dot{w}_0.B)).$$

Theorem 4.9. *ρ_t is a Poisson open embedding.*

Proof. By Corollary 4.2, the action map σ of G on $G \times_B B$ restricts to give a Poisson map

$$\sigma:\quad (G^{1,w^{-1}w_0},\ \pi_G) \times (X^1_{t,w_0},\ \pi) \longrightarrow (X_{t,w},\ \pi). \tag{4.11}$$

Since $\tilde{J}_t$ is Poisson, it follows that ρ_t is Poisson and the image is in $X_{t,w}$ by Lemma 4.1. The morphism $U \times U \to X_t$ given by $(m,n) \mapsto [mn\dot{w}_0, \xi(n)]$ is easily seen to be an embedding, since ξ is injective. It follows that ρ_t is injective, and it is an open embedding by the same argument as in [9], proof of Theorem 1.2, p. 358. □

Remark 4.10. In a later paper, we will use the open embedding ρ_t to find log-canonical coordinates on an open subset of $X_{t,w}$, and study the associated cluster algebra.

5. Appendix

5.1. **Poisson Lie groups.** In this appendix, we recall some general facts on Poisson Lie groups that are used in the construction of the Poisson structure π_0 on G in §2. Some of the omitted details in this section can be found in [1] and [16].

Recall that a Poisson bi-vector field π_G on a Lie group G is said to be *multiplicative* if the map $m : G \times G \to G : (g_1, g_2) \mapsto g_1g_2$ is a Poisson map with respect to π_G. A *Poisson Lie group* is a Lie group G with a multiplicative Poisson bi-vector field π_G. An action $\sigma : G \times P \to P$ of a Poisson Lie group (G, π_G) on a Poisson manifold (P, π_P) is said to be Poisson if σ is a Poisson map.

If (G, π_G) is a Poisson Lie group, then $\pi_G(e) = 0$, where $e \in G$ is the identity element. The linearization of π_G at e is the linear map $\delta_{\mathfrak{g}} : \mathfrak{g} \to \wedge^2\mathfrak{g}$ given by $\delta_{\mathfrak{g}}(x) = [\tilde{x}, \pi_G](e)$, where for $x \in \mathfrak{g}$, $\tilde{x}$ is any vector field on G with $\tilde{x}(e) = x$, and $[\tilde{x}, \pi_G]$ is the Lie derivative of π_G by $\tilde{x}$.

Two Poisson Lie groups (G, π_G) and (G^*, π_{G^*}) are said to be *dual* to each other if their Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ are dual to each other and if the dual map of $\delta_{\mathfrak{g}} : \mathfrak{g} \to \wedge^2\mathfrak{g}$ is the Lie bracket on $\mathfrak{g}^*$ and the dual map of $\delta_{\mathfrak{g}^*} : \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ is the Lie bracket on $\mathfrak{g}$.

One important class of Poisson Lie groups is constructed from *Manin triples.* Recall that a Manin triple is a quadruple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*, \langle\ ,\ \rangle)$, where $\mathfrak{d}$ is an even dimensional Lie algebra, $\langle\ ,\ \rangle$ is a symmetric non-degenerate invariant bilinear form on $\mathfrak{d}$, $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras of $\mathfrak{d}$, both maximally isotropic with respect to $\langle\ ,\ \rangle$, and

$\mathfrak{g} \cap \mathfrak{g}^* = 0$. The bilinear form $\langle\ ,\ \rangle$ gives rise to a non-degenerate pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, so $\mathfrak{g}^*$ can indeed be regarded as the dual space of $\mathfrak{g}$.

Assume that $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*, \langle\ ,\ \rangle)$ is a Manin triple. Let $\{x_i\}$ be a basis of $\mathfrak{g}$ and let $\{\xi_i\}$ be the dual basis of $\mathfrak{g}^*$. Let

$$\Lambda = \frac{1}{2} \sum_i \xi_i \wedge x_i \in \wedge^2 \mathfrak{d}.$$

Then Λ is independent of the choices of the bases, and the Schouten bracket $[\Lambda, \Lambda] \in \wedge^3 \mathfrak{d}$ is ad-invariant. Let D be a connected Lie group with Lie algebra $\mathfrak{d}$. Define the bi-vector fields $\pi_D^{\pm}$ on D by

$$\pi_D^{\pm} = \Lambda^R \pm \Lambda^L,$$

where Λ^R and Λ^L are respectively the right and left invariant bi-vector fields on G with values Λ at the identity. Then π_D^- and π_D^+ are Poisson structures on D (see [16], Proposition 3.4.1 for a proof that π_D^- is Poisson, and use the fact that left and right-invariant vector fields commute to see $[\pi_D^+, \pi_D^+] = [\pi_D^-, \pi_D^-]$, so π_D^+ is Poisson). Let G and G^* be the connected subgroups of D with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Then it is easy to check that both G and G^* are Poisson submanifolds of (D, π_D^-). Set

$$\pi_G = \pi_D^-|_G \quad \text{and} \quad \pi_{G^*} = -\pi_D^-|_{G^*}.$$

Then (G, π_G) and (G^*, π_{G^*}) is a pair of dual Poisson Lie groups. Moreover, (G, π_G) and $(G^*, -\pi_{G^*})$ are Poisson subgroups of (D, π_D^-) (this follows using Remark 2.3.5 in [16]). Thus, any Poisson action of (D, π_D^-) on a Poisson manifold restricts to Poisson actions of (G, π_G) and $(G^*, -\pi_{G^*})$.

The following Lemma 5.1 is immediate from definitions.

Lemma 5.1. *The two actions*

$$(D, \pi_D^-) \times (D, \pi_D^+) \longrightarrow (D, \pi_D^+) : \quad (d_1, d_2) \longmapsto d_1 d_2$$
$$(D, \pi_D^+) \times (D, -\pi_D^-) \longrightarrow (D, \pi_D^+) : \quad (d_1, d_2) \longmapsto d_1 d_2$$

are Poisson.

A proof of the following Proposition 5.2 can be found in [1], and follows from results of [3] and Lu-Yakimov-2. We give an outline of the proof for completeness.

Proposition 5.2. *Assume that G is a closed subgroup of D, and let $\eta : D \to D/G$ be the natural projection. Then*

$$\pi_0 := \eta(\pi_D^{\pm}) = \eta(\Lambda^R)$$

is a well-defined Poisson structure on D/G. Moreover, the actions

$$(G, \pi_G) \times (D/G,\ \pi_0) \longrightarrow (D/G,\ \pi_0) : \quad (g, dG) \longmapsto gdG,$$
$$(G^*, -\pi_{G^*}) \times (D/G,\ \pi_0) \longrightarrow (D/G,\ \pi_0) : \quad (u, dG) \longmapsto udG$$

are Poisson, and symplectic leaves of π_0 in D/G are precisely the connected components of non-empty intersections of G and G^* orbits in D/G.

Proof. The element $\Lambda \in \wedge^2 \mathfrak{d}$ is mapped to 0 under the projection $\mathfrak{d} \to \mathfrak{d}/\mathfrak{g}$. Thus $\eta(\Lambda^L) = 0$. Since $\eta(\Lambda^R)$ is clearly well-defined, π_0 is well-defined. Now the Poisson action of (D, π_D^-) on (D, π_D^+) by left multiplication restricts to give Poisson actions of (G, π_G) and $(G^*, -\pi_{G^*})$, which clearly descend to give Poisson actions on $(D/G,\ \pi_0)$.

Note that since $\mathfrak{g}+\mathfrak{g}^*_{\rm st}=\mathfrak{d}$, for any $\underline{d}=dG\in D/G$, where $d\in D$, the G orbit $G.\underline{d}$ and the G^* orbit $G^*\!.\underline{d}$ intersect transversally at $\underline{d}$. In particular, $T_{\underline{d}}((G.\underline{d})\cap(G^*\!.\underline{d}))=T_{\underline{d}}(G.\underline{d})\cap T_{\underline{d}}(G^*\!.\underline{d}))$. Thus, to prove the statement about the symplectic leaves of π_0, it is enough to check that $T_{\underline{d}}(G.\underline{d})\cap T_{\underline{d}}(G^*\!.\underline{d}))$ coincides with the tangent space to the symplectic leaf of π_0 at $\underline{d}$. To this end, identify $T_{\underline{d}}(D/G)\cong\mathfrak{d}/\mathrm{Ad}_d\mathfrak{g}$. Then $\pi_0(\underline{d})$ becomes the element $p_d(\Lambda)\in\wedge^2(\mathfrak{d}/\mathrm{Ad}_d\mathfrak{g})$, where $p_d:\mathfrak{d}\to\mathfrak{d}/\mathrm{Ad}_d\mathfrak{g}$ is the projection. Let $\tilde{\Lambda}:\mathfrak{d}\to\mathfrak{d}$ be the map

$$\tilde{\Lambda}(x+\xi)=\frac{1}{2}\sum_{i=1}^{n}(\langle x+\xi,\ \xi_i\rangle x_i-\langle x+\xi,\ x_i\rangle\xi_i)=\frac{1}{2}(x-\xi),\qquad x\in\mathfrak{g},\xi\in\mathfrak{g}^*.$$

Let $S_{\underline{d}}$ be the symplectic leaf of π_0 through $\underline{d}$. Then by definition,

$$T_{\underline{d}}S_{\underline{d}}=p_d\left(\tilde{\Lambda}(\mathrm{Ad}_d\mathfrak{g})\right)\subset\mathfrak{d}/\mathrm{Ad}_d\mathfrak{g}.$$

For any $x+\xi\in\mathrm{Ad}_d\mathfrak{g}$,

$$\tilde{\Lambda}(x+\xi)=\frac{1}{2}(x-\xi)=x-\frac{1}{2}(x+\xi)=-\xi+\frac{1}{2}(x+\xi).\tag{5.1}$$

It follows that $p_d(\tilde{\Lambda}(\mathrm{Ad}_d\mathfrak{g}))=p_d(\mathfrak{g})\cap p_d(\mathfrak{g}^*)\cong T_{\underline{d}}(G.\underline{d})\cap T_{\underline{d}}(G^*\!.\underline{d}))$.

□

The following Lemma 5.3 on the symplectic leaves of π_D^+ in D is proved in [1]. We give a slightly different proof here for completeness. Moreover, (5.3) and (5.4) in our proof of Lemma 5.3 are used in the proof of Lemma 3.9.

Lemma 5.3. *Symplectic leaves of π_D^+ in D are precisely the connected components of non-empty intersections of (G,G^*)-double cosets and (G^*,G)-double cosets in D.*

Proof. Let $d\in D$. Then

$$T_d(G^*dG)+T_d(GdG^*)=r_d\mathfrak{g}^*+l_d\mathfrak{g}+r_d\mathfrak{g}+l_d\mathfrak{g}^*=r_d\mathfrak{d}=T_dD.$$

Hence G^*dG and GdG^* intersect transversally at d. Let Σ be the symplectic leaf of π_D^+ through d. It is enough to show that

$$T_d\Sigma=T_d(G^*dG)\cap T_d(GdG^*).$$

Let $\tilde{\pi}_D^+:T^*D\to TD$ be the bundle map defined by π_D^+ (see (1.1)). Identify $\mathfrak{d}$ with $\mathfrak{d}^*$ via $\langle\ ,\ \rangle$, and for $x\in\mathfrak{g}$ and $\xi\in\mathfrak{g}^*$, let $\alpha_{x+\xi}$ be the right invariant 1-form on D with value $x+\xi$ at the identity element of D. From definitions and (5.1),

$$\tilde{\pi}_D^+(\alpha_{x+\xi})(d)=r_d\tilde{\Lambda}(x+\xi)+l_d\tilde{\Lambda}(\mathrm{Ad}_d^{-1}(x+\xi))\tag{5.2}$$

$$=-r_d(\xi)+l_d\left(p_{\mathfrak{g}}\mathrm{Ad}_d^{-1}(x+\xi)\right)\tag{5.3}$$

$$=r_d(x)-l_d\left(p_{\mathfrak{g}^*}\mathrm{Ad}_d^{-1}(x+\xi)\right),\tag{5.4}$$

where $p_{\mathfrak{g}}:\mathfrak{d}\to\mathfrak{g}$ and $p_{\mathfrak{g}^*}:\mathfrak{d}\to\mathfrak{g}^*$ are the projections with respect to the decomposition $\mathfrak{d}=\mathfrak{g}+\mathfrak{g}^*$. Thus $T_d\Sigma\subset T_d(G^*dG)\cap T_d(GdG^*)$. Conversely, if $v_d\in T_d(G^*dG)\cap T_d(GdG^*)$, then

$$v_d=-r_d(\xi)+l_d(x_1)=r_d(x)-l_d(\xi_1)$$

for some $x,x_1\in\mathfrak{g}$ and $\xi,\xi_1\in\mathfrak{g}^*$. Then $x_1+\xi_1=\mathrm{Ad}_d^{-1}(x+\xi)$, so $v_d=\tilde{\pi}_D^+(\alpha_{x+\xi})(d)\in T_d\Sigma$. Hence $T_d\Sigma=T_d(G^*dG)\cap T_d(GdG^*)$.

□

5.2. **Coisotropic reduction.** In this section, we prove a fact, Proposition 5.6, that is used in the study of the Poisson structure π on $G \times_B B$ in §3. Since we can not find the statement in the literature, we give a proof here for completeness.

By definition, a *Poisson vector space* is a pair (V, π), where V is a vector space and $\pi \in \wedge^2 V$. Assume that V is finite dimenaional. For such a pair (V, π), let $\tilde{\pi}$ be the linear map

$$\tilde{\pi}: \quad V^* \longrightarrow V: \quad (\tilde{\pi}(\xi), \, \eta) = \pi(\xi, \eta), \qquad \xi, \eta \in V^*$$

and set $V_\pi = \tilde{\pi}(V^*) \subset V$. By definition, a subspace V_1 of V is a Poisson vector subspace of (V, π) if $V_1 \supset V_\pi$. This is the case if and only if $\pi \in \wedge^2 V_1$, so that (V_1, π) is a Poisson vector space.

Recall [26] that a subspace U of V is said to be coisotropic with respect to π if $\tilde{\pi}(U^\perp) \subset U$, where

$$U^\perp = \{\xi \in V^*: \quad (\xi, x) = 0, \; \forall x \in U\}.$$

Remark 5.4. It is easy to check that U is coisotropic if and only if $\pi \in U \wedge V \subset \wedge^2 V$.

If (V, π) is a Poisson vector space, and $p: V \to V/W$ is the projection, then $(V/W, p(\pi))$ is a Poisson vector space.

Let U be a coisotropic subspace of (V, π), and let $\phi: V \to V/\tilde{\pi}(U^\perp)$ be the projection. Set $\varpi = \phi(\pi)$. $(V/\tilde{\pi}(U^\perp), \, \varpi)$ is a Poisson vector space.

Lemma 5.5. *Let U be a coisotropic subspace of (V, π), let $\phi: V \to V/\tilde{\pi}(U^\perp)$ be the projection, and set $\varpi = \phi(\pi)$.*

$$(V/\tilde{\pi}(U^\perp))_\varpi = \phi(U \cap V_\pi) \subset U/\tilde{\pi}(U^\perp) \subset V/\tilde{\pi}(U^\perp). \tag{5.5}$$

In particular, $(U/\tilde{\pi}(U^\perp), \, \varpi)$ is a Poisson vector subspace of $(V/\tilde{\pi}(U^\perp), \, \varpi)$.

Proof. The map $\phi^*: (V/\tilde{\pi}(U^\perp))^* \to V^*$ gives an isomorphism between $(V/\tilde{\pi}(U^\perp))^*$ and $\tilde{\pi}^{-1}(U)$. Thus,

$$\tilde{\pi} \circ \phi^*(V/\tilde{\pi}(U^\perp)^*) = \tilde{\pi}(\tilde{\pi}^{-1}(U)) = U \cap \tilde{\pi}(V^*).$$

Using the identity $\phi \circ \tilde{\pi} \circ \phi^* = \tilde{\varpi}$, we obtain $(V/\tilde{\pi}(U^\perp))_\varpi = \phi(U \cap V_\pi)$. The remaining statements follow easily. □

Let (P, π_P) be a Poisson manifold. A submanifold $Q \subset P$ is said to be coisotropic if $\tilde{\pi}_P((T_q Q)^\perp) \subset T_q Q$ for every $q \in Q$, where $(T_q Q)^\perp$ is the conormal bundle of Q in P and $\tilde{\pi}_P$ is the bundle map from T^*P to TP given in (1.1). In this case, $C_Q := \tilde{\pi}_P((T_q Q)^\perp)$ is called the *characteristic distribution* of π_P on Q.

Proposition 5.6. *Assume*

1) (P, π_P) is a Poisson manifold with a free right Poisson action by (B, π_B) and P/B is a manifold,

2) Q is a B-invariant coisotropic submanifold of (P, π_P) and the characteristic distribution of π on Q is the same as the distribution defined by the B-action,

3) For each $q \in Q$, Q intersects with the symplectic leaf S_q of π_P through q cleanly, i.e., $Q \cap S_q$ is a submanifold and $T_q(Q \cap S_q) = T_q Q \cap T_q S_q$.

Let $\phi: P \to P/B$ be the projection. Then, $\phi(\pi_P)$ is a well-defined Poisson structure on P/B, and Q/B is a Poisson submanifold of P/B with respect to $\phi(\pi_P)$.

Moreover, for each $q \in Q$, the symplectic leaf of $\phi(\pi_P)$ in Q/B through $\phi(q)$ is the connected component of $\phi(Q \cap S_q)$ through $\phi(q)$.

Proof. By the definition of Poisson action, $\phi(\pi_P)$ is a well-defined bi-vector on P/B, and it is easy to check it is Poisson. By the last statement of Lemma 5.5, Q/B is a Poisson submanifold of $(P/B, \phi(\pi_P))$. (5.5) of Lemma 5.5 gives the assertion on symplectic leaves. □

5.3. **Singularities of intersections of Bruhat cells and Steinberg fibers.** For an affine variety X with ring of regular functions $O(X)$ and $g_1, \ldots, g_k \in O(X)$, let $V(g_1, \ldots, g_k)$ denote the common vanishing set of $g_1, \ldots, g_k$, and let $(g_1, \ldots, g_k)$ denote the ideal in $O(X)$ generated by $g_1, \ldots, g_k$. If $Y \subset X$ is Zariski closed, let $I(Y)$ be the ideal of regular functions vanishing on Y.

Proposition 5.7. *Let C be a conjugacy class in G and let $w \in W$. Assume $C \cap BwB_-$ is nonempty. Then $C \cap BwB_-$ is smooth and irreducible, and* $\dim(C \cap BwB_-) = \dim(C) - l(w)$.

Proof. This follows from Proposition 4.10 of [8], taking $X = G$, $D = G \times G$, A as the diagonal in D, and $C = B \times B_-$. □

Assume G is simply connected and let $\chi : G \to \mathbb{C}^r$ be the Steinberg map with Steinberg fiber $F_z = \chi^{-1}(z)$ for $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$. For a Bruhat variety $\overline{BwB_-}$, let

$$f_i = \chi_i|_{\overline{BwB_-}} - z_i.$$

Proposition 5.8. *(1)* $F_z \cap \overline{BwB_-} = V(f_1, \ldots, f_r)$.
(2) $F_z \cap \overline{BwB_-}$ is nonempty and irreducible
(3) $\dim(F_z \cap \overline{BwB_-}) = d = \dim(G) - r - l(w)$.

Proof. Since $F_z = V(\chi_1 - z_1, \ldots, \chi_r - z_r)$, (1) is clear. By Lemma 2.15, $F_z \cap \overline{BwB_-}$ is nonempty. Let V be an irreducible component of $V(f_1, \ldots, f_r)$ and note that $\dim(V) \geq \dim(G) - r - l(w)$. Let $F_z = \cup_{i=1}^n C_{z_i}$ be the decomposition of F_z into conjugacy classes with $C_{z_1} = R_z$ the unique regular conjugacy class in F_z. Then $F_z \cap \overline{BwB_-} = \cup_{i=1,\ldots,n, y \geq w} C_{z_i} \cap ByB_-$. If $i > 1$ or $y > w$, then by Proposition 5.7,

$$\dim(C_{z_i} \cap ByB_-) = \dim(C_{z_i}) - l(y) < \dim(R_z) - l(w) = d.$$

It follows easily that $V(f_1, \ldots, f_r)$ is irreducible. □

Lemma 5.9. *$\overline{BwB_-}$ is Cohen-Macaulay for all $w \in W$.*

Proof. By a Theorem of Ramanathan, $\overline{B_-wB_-}/B_-$ is Cohen-Macaulay in G/B_- [21]. The result now follows easily using the fact that the smooth morphism $G \to G/B_-$ is a locally trivial bundle in the Zariski topology and using the isomorphism $\overline{B_-wB_-} \cong \overline{Bw_0wB_-}$ given by left multiplication by $\dot{w}_0$. □

Lemma 5.10. *(see* [14], *Lemma 7.1) Let Y be an irreducible Cohen-Macaulay affine variety of dimension n, and let $f_1, \ldots, f_r \in O(Y)$. Let $X = V(f_1, \ldots, f_r)$. Suppose*
(1) X is irreducible and
(2) there is a smooth point $y \in X$ such that $(df)_1(y), \ldots, (df)_r(y)$ are linearly independent.

Then $\dim(X) = n - r$ *and the ideal* $(f_1, \ldots, f_r) = I(X)$.

Remark 5.11. Our statement is more general than the statement in [14]. The proof is identical, once we recall a basic fact about Cohen-Macaulay varieties. The ideal $(f_1, \ldots, f_r) = Q_1 \cap \cdots \cap Q_s$ has a minimal primary decomposition. The Cohen-Macaulay condition ensures that if $P_i = \sqrt{Q_i}$ for $i = 1, \ldots, s$, the varieties $V(P_i)$ all have the same dimension (by Theorem 17.6 in [20]).

Theorem 5.12. *$(f_1, \ldots, f_r)$ is the ideal of the irreducible variety $F_z \cap \overline{BwB_-}$ in $\overline{BwB_-}$. Moreover, $F_z \cap \overline{BwB_-}$ is Cohen-Macaulay.*

Proof. By Lemma 5.9, $\overline{BwB_-}$ is Cohen-Macaulay. By Proposition 5.8, $F_z \cap \overline{BwB_-}$ is irreducible. Recall that $(\chi_1 - z_1, \ldots, \chi_r - z_r)$ is the ideal of F_z and R_z is the smooth locus of F_z ([13], Theorem 4.24). In particular,

$$R_z = \{y \in F_z : (d\chi)_1(y), \ldots, (d\chi)_r(y) \text{ are linearly independent on } T_y(G)\}$$

and $T_y(R_z)$ is defined in $T_y(G)$ by the vanishing of $(d\chi)_1(y), \ldots, (d\chi)_r(y)$.

Since R_z and BwB_- are smooth locally closed subvarieties of G and R_z meets BwB_- transversally, $R_z \cap BwB_-$ is smooth and locally closed in $F_z \cap \overline{BwB_-}$. Let $y \in R_z \cap BwB_-$ and let λ be a nonzero covector in the span of $(d\chi)_1(y), \ldots, (d\chi)_r(y)$. Since $T_y(R_z) + T_y(BwB_-) = T_y(G)$, it follows that λ is nonzero on $T_y(BwB_-)$. Thus, the restrictions $(df)_1(y), \ldots, (df)_r(y)$ of $(d\chi)_1(y), \ldots, (d\chi)_r(y)$ to $T_y(BwB_-)$ are linearly independent. Since $T_y(BwB_-) = T_y(\overline{BwB_-})$, we can apply Lemma 5.10 to deduce the first assertion. Since $\overline{BwB_-}$ is Cohen-Macaulay, $F_z \cap \overline{BwB_-}$ is Cohen-Macaulay using [23], Corollary, page 65. □

Proposition 5.13. *(1) Let $\overline{BwB_-}_{\mathrm{ns}}$ be the smooth locus of $\overline{BwB_-}$. Then $R_z \cap \overline{BwB_-}_{\mathrm{ns}}$ is the smooth locus of $F_z \cap \overline{BwB_-}_{\mathrm{ns}}$.*
(2) The singular locus of $F_z \cap \overline{BwB_-}$ has codimension at least 2.

Proof. By Theorem 5.12, the ideal sheaf of $F_z \cap \overline{BwB_-}_{\mathrm{ns}}$ is generated by $f_1, \ldots, f_r$. By the Jacobian criterion, the smooth locus of $F_z \cap \overline{BwB_-}_{\mathrm{ns}}$ is the set of points $y \in F_z \cap \overline{BwB_-}_{\mathrm{ns}}$ where $(df)_1(y), \ldots, (df)_r(y)$ are linearly independent on $T_y(\overline{BwB_-}_{\mathrm{ns}})$. Let $y \in R_z \cap \overline{BwB_-}_{\mathrm{ns}}$ be in BvB_-. Using transversality as in the proof of Theorem 5.12, it follows that $(df)_1(y), \ldots, (df)_r(y)$ are linearly independent on $T_y(BvB_-)$, so they are linearly independent on $T_y(\overline{BwB_-}_{\mathrm{ns}}) \supset T_y(BvB_-)$. Conversely, if $y \in F_z - R_z$, $(d\chi)_1(y), \ldots, (d\chi)_r(y)$ are linearly dependent in $T_y(G)$. As a consequence, their restrictions $(df)_1(y), \ldots, (df)_r(y)$ are linearly dependent on $T_y(\overline{BwB_-}_{\mathrm{ns}})$, which gives (1). For (2), note that if y is a singular point of $F_z \cap \overline{BwB_-}$, then either y is a singular point of $\overline{BwB_-}$ or y is a singular point of $F_z \cap \overline{BwB_-}_{\mathrm{ns}}$. Since the singular set of $\overline{BwB_-}$ has codimension at least two [2], it suffices to show that the singular set of $F_z \cap \overline{BwB_-}_{\mathrm{ns}}$ has codimension at least two. Let $F_z = \cup_{i=1}^{n} C_{z_i}$ be the decomposition into conjugacy classes. By (1), the singular set of $F_z \cap \overline{BwB_-}_{\mathrm{ns}}$ is contained in

$$\cup_{v \geq z, i \geq 2} C_{z_i} \cap BvB_-.$$

By [13], Theorem 4.24, if $i \geq 2$, $\dim(C_{z_i}) \leq \dim(R_z) - 2$. Since $\dim(C_{z_i} \cap BvB_-) = \dim(C_{z_i}) - l(y)$ by Proposition 5.7, it follows that if $i \geq 2$,

$$\dim(C_{z_i} \cap BvB_-) \leq \dim(R_z \cap BvB_-) - 2 \leq \dim(F_z \cap \overline{BwB_-}) - 2.$$

(2) follows. □

Theorem 5.14. *$F_z \cap \overline{BwB_-}$ is normal.*

Proof. Since $F_z \cap \overline{BwB_-}$ is Cohen-Macaulay by Theorem 5.12, condition S_2 of Serre is satisfied (see [20], p.183). Part (2) of Proposition 5.13 is equivalent to condition R_1 of Serre, so the theorem follows using Serre's normality criterion ([20], Theorem 23.8). □

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