CHARACTERISTIC CYCLES FOR THE LOOP GRASSMANNIAN AND NILPOTENT ORBITS

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0. Introduction

Let $\mathcal{F}$ be a perverse sheaf on a complex manifold X, constructible with respect to a Whitney stratification $\mathcal{S}$. A basic problem in microlocal geometry is to compute the characteristic cycle,

$$CC(\mathcal{F}) = \sum_{\alpha \in \mathcal{S}} c_\alpha(\mathcal{F}) \cdot \overline{T^*_\alpha(X)},$$

a linear combination of closures of conormal bundles to submanifolds of X. Intuitively, the microlocal multiplicities $c_\alpha(\mathcal{F})$ measure the singularity of $\mathcal{F}$ at α. In settings related to representation theory, a group G acts on X, $\mathcal{F}$ is G-equivariant and the microlocal multiplicities play a significant but only partially understood role in representation theory (see [Ro],[SV],[ABV], [KaSa] e.g.).

In this paper, we compute microlocal multiplicities in certain cases of interest in representation theory. Let G be a connected reductive group with loop group LG and let P be the subgroup of LG consisting of loops with positive Fourier coefficients. Then P-orbits $\mathcal{G}_\lambda$ on the loop Grassmannian $LG/P = \mathcal{G}$ correspond to the irreducible representations $L(\lambda)$ of the dual reductive group $\check{G}$. For dominant weights μ and λ of a torus of $\check{G}$, let $m_\mu\,(L(\lambda))$ denote the multiplicity of μ in $L(\lambda)$. Let us embed the orbit closure $\overline{\mathcal{G}_\lambda}$ into a finite dimensional manifold Z.

0.1. Theorem.

(a) Let $i : \mathcal{G}_\lambda \to Z$ be the inclusion and let $i_!(\mathbb{C}_\lambda)$ be the extension by 0 of the constant sheaf on $\mathcal{O}_\lambda$. Then $CC(i_!(\mathbb{C}_\lambda)[\dim \mathcal{G}_\lambda]) = \overline{T^*_{\mathcal{G}_\lambda} Z}$.

(b) The characteristic cycle of the intersection cohomology sheaf of $\overline{\mathcal{G}_\lambda}$ is given by the character of $L(\lambda)$,

$$CC(IC_\lambda) = \sum_{\mathcal{G}_\mu \subseteq \overline{\mathcal{G}_\lambda}} m_\mu(L(\lambda)) \cdot \overline{T^*_{\mathcal{G}_\mu} Z}.$$

We also consider the nilpotent cone in a semisimple Lie algebra.

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0.2. Theorem. *Let $\mathfrak{g} = \mathfrak{sl}(n)$ and let α and β be distinct nilpotent orbits and $i : \alpha \hookrightarrow \mathfrak{g}$. Then*

(a) $CC(i_!(\mathbb{C}_\beta)) = \overline{T^*_\beta(\mathfrak{g})}$.

(b) The microlocal multiplicity $c_\alpha(IC(\beta))$ of the intesection homology sheaf of $\bar\beta$ along the orbit α is the multiplicity of the Springer representation $V(\beta)$ of the permutation group S_n in the cohomology $H^*(\mathcal{B}_e)$ of the Springer fiber $\mathcal{B}_e$ at $e \in \alpha$ ($\mathcal{B}$ is the flag variety of $\mathfrak{g}$ and $\mathcal{B}_e$ is the zero set of the vector field given by e).

We also restate this result in terms of character sheaves on GL_n. Theorem 0.2 was later proved independently by Grinberg by a different method from ours [Gr]. On the other hand, our description of the microlocal fundamental group of P-orbits in $\mathcal{G}$ (as quotients of certain braid groups) is inspired by the analogous result of Grinberg for nilpotent cones. The analogue of theorem 0.2 is false for other simple Lie algebras. In general we reduce the computation of characteristic cycles to the case of distinguished orbits. We observe various consequences of this and we also consider subregular distinguished orbits.

We consider these results as a step towards microlocal descriptions of the categories of character sheaves on G and P-equivariant sheaves on $\mathcal{G}$ (for an arbitrary coefficient ring, see [MV2]). Our approach is based on the study of the Euler obstruction, a local invariant of singularities introduced in different forms by MacPherson, Dubson, and Kashiwara. The Euler obstruction is the Euler characteristic of the complex link. Our method is to construct an action of a torus on the complex link. This is possible in the cases when the conormal space to an orbit can be identified with the Lie algebra of the algebraic group which stabilizes a point in the orbit, provided that the stabilizer contains a nontrivial torus. We note that other approaches to the computation of characteristic cycles are different-either they exploit the existence of a small resolution [BFL], an elaborate inductive argument [BF] or a special form of the defining equation in the case of toric varieties ([GKZ], [Br]).

The structure of our paper is as follows. In the first section, we recall the definition of the Euler obstruction, the Dubson-Kashiwara index formula which relates the Euler obstruction to microlocal multiplicities, and recall some standard results concerning torus actions. In the second section, we prove Theorem 0.1 and in the third section, we prove Theorem 0.2. We apply the results of the third section in the fourth section to compute characteristic cycles of character sheaves.

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1. Index formula and reduction to fixed points

In this section, we review the index formula of Dubson and Kashiwara and observe that it permits the calculation of characteristic cycles via localization to the fixed points.

1.1. Euler obstructions. Let $\mathcal{S}$ be a Whitney stratification of a complex manifold X ([KaSc]). Let $D_{\mathcal{S}}(X)$ denote the derived category of constructible sheaves on X and let $\mathcal{F}$ be an object in $D_{\mathcal{S}}(X)$. Any stratum $\alpha \in \mathcal{S}$ defines two numerical invariants of $\mathcal{F}$, the Euler characteristic $\chi_\alpha(\mathcal{F})$ of the stalk of $\mathcal{F}$ at any point in α , and the microlocal multiplicity $c_\alpha(\mathcal{F})$ of $\mathcal{F}$ along α, which appears in the characteristic cycle

$$CC(\mathcal{F}) = \sum_{\alpha \in \mathcal{S}} c_\alpha(\mathcal{F}) \cdot \overline{T^*_\alpha(X)}$$

of $\mathcal{F}$ (see [KaSc]). Here $T^*_\alpha(X)$ denotes the conormal bundle to the stratum α.

For any pair of strata α and β in $\mathcal{S}$, the Euler obstruction $e_{\alpha,\beta}$ is defined by $e_{\alpha,\beta} = c_\alpha(\mathbb{C}_\beta)$, where we denote by $\mathbb{C}_\beta$ the constant sheaf on β extended by zero to X.

A covector $\xi \in T^*_\alpha(X)$ is called $\mathcal{S}$-generic (or $\mathcal{S}$-regular) if it lies in $T^*_\alpha(X)^r \stackrel{\text{def}}{=} T^*_\alpha(X) - \bigcup_{\alpha \neq \beta \in \mathcal{S}} \overline{T^*_\beta(X)}$. The set of generic elements $T^*_\alpha(X)^r$ is open and dense in $T^*_\alpha(X)$ and its fundamental group is called the microlocal fundamental group of the stratum α.

1.2 Theorem (see [Du],[Ka1], or [Gi1, 8.2]). *(a) On the the Grothendieck group of $\mathcal{S}$-contructible complexes*

$$c_\alpha = \sum_{\alpha \subset \overline{\beta}} e_{\alpha,\beta} \cdot \chi_\beta, \quad \alpha \in \mathcal{S}.$$

(b) One has $e_{\alpha,\alpha} = (-1)^{dim(\alpha)}$ and $e_{\alpha,\beta} = 0$ if $\alpha \not\subset \overline{\beta}$. If $\alpha \subset \partial\overline{\beta}$, we choose (i) a normal slice N in $(X, \mathcal{S})$ to α at a point $x \in \alpha$, (ii) a holomorphic function ϕ on N vanishing at x and such that $d_x\phi \in T^*_x(N)$ is $\mathcal{S}$-generic, (iii) a small ball B around x in X and lastly, (iv) a small enough $t \in \mathbb{C}$. Then

$$e_{\alpha,\beta} = (-1)^{dim(\alpha)+1} \chi_c \left(\beta \cap \phi^{-1}(t) \cap B\right),$$

is up to sign the Euler characteristic of compactly supported cohomology of the intersection of β with a nearby hyperplane $\phi^{-1}(t)$, near α and normally to α (we call the space $\beta \cap \phi^{-1}(t) \cap B$ the complex link of the strata α and β).

1.3. Euler characteristics and localization to the fixed points. We recall the following well-known fact which will be exploited throughout the paper.

1.4. Proposition. *Let X be a paracompact space with finite cohomological dimension with the action of a compact torus T. Then*

$$\chi_c(X) = \chi_c(X^T),$$

where X^T is the fixed point set. In particular, if X^T is empty, then $\chi_c(X) = 0$.

This may be proved via the localization formula of Borel for equivariant cohomology, $H_{T,c}(X) \cong H_{T,c}(X^T)$ after localizing (see [Hs], IV.1 for the case of ordinary cohomology. The proof for compactly supported cohomology is similar, or one may find a reference adequate for this paper in [EM], appendix B among others).

1.5. Proposition [St, 2.13]. *Let H be an algebraic group with Lie algebra $\mathfrak{h}$. The subset $\mathfrak{h}_s \subseteq \mathfrak{h}$, consisting of all elements x for which there is a maximal torus of H stabilizing x, contains an open dense set.*

2. Vanishing of Euler obstructions for the loop Grassmannian

In this section we will prove that all nontrivial Euler obstructions are zero for the loop Grassmannian.

2.0. The loop Grassmannian $\mathcal{G}$ (see [Lu2]). Let $\mathcal{K} = \mathbb{C}((t))$ denote formal Laurent series with ring of integers $\mathcal{O} = \mathbb{C}[[t]]$, the ring of formal Taylor series. For G a complex connected reductive group, then $LG = G(\mathcal{K})$ is the associated loop group with parabolic subgroup $P = G(\mathcal{O})$. The loop Grassmannian of G is the homogeneous space $\mathcal{G} = LG/P$. We denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{g}_\mathcal{K}$ and $\mathfrak{p} = \mathfrak{g}_\mathcal{O}$ the Lie algebras given by extending scalars.

For convenience we choose a maximal torus and a Borel subgroup $T \subseteq B \subseteq G$. Denote by $X_*(T) \stackrel{\text{def}}{=} \text{Hom}(G_m, T)$ the cocharacters of T. An element $\lambda \in X_*(T)$ defines a homomorphism $\lambda : \mathcal{K}^* \to T(\mathcal{K})$ and therefore also an element $\lambda(z) \in T(\mathcal{K}) \subseteq LG$. Let L_λ be its image in $\mathcal{G}$ and $\mathcal{G}_\lambda \stackrel{\text{def}}{=} P{\cdot}L_\lambda$, the corresponding P–orbit in $\mathcal{G}$. In this way, P-orbits in $\mathcal{G}$ are parametrized by B-dominant cocharacters $X_*(T)_+ \subseteq X_*(T)$ and more generally, for $\lambda,\ \mu \in X_*(T)$, $\mathcal{G}_\lambda = \mathcal{G}_\mu$ if and only if λ and μ lie in the same orbit of the Weyl group of G with respect to T. The fixed point set $\mathcal{G}^T$ for the T-action on $\mathcal{G}$ is a bijective image of $X_*(T)$ under the map $\lambda \mapsto L_\lambda$.

The loop Grassmannian $\mathcal{G}$ is an ind-variety, an increasing union of the finite dimensional projective varieties given by the closures of P-orbits. For dominant λ and μ, $\mathcal{G}_\lambda \subset \overline{\mathcal{G}_\mu}$ if and only if $\mu \geq \lambda$, i.e., $\lambda - \mu$ is a sum of positive coroots.

2.1. Formal computation of conormals to P-orbits. Consider the "$\mathfrak{g}_\mathcal{K}$-stabilizer bundle over $\mathcal{G}$" $LG \underset{P}{\times} \mathfrak{p}$ with the fiber at gL_0 ($g \in LG$) equal ${}^g\mathfrak{p}$. We will denote $T\mathcal{G} \stackrel{\text{def}}{=} (\mathfrak{g}_\mathcal{K} \times \mathcal{G})/LG \underset{P}{\times} \mathfrak{p}$ and $T^*\mathcal{G} \stackrel{\text{def}}{=} LG \underset{P}{\times} \mathfrak{p}$. Let $\Lambda \subseteq T^*\mathcal{G}$ be the $\mathfrak{p}$-stabilizer subscheme over $\mathcal{G}$, i.e., $\Lambda_{g\cdot L_0} = {}^g\mathfrak{p} \cap \mathfrak{p},\ g \in LG$, and denote by Λ_λ the restriction to the P-orbit $\mathcal{G}_\lambda$. If $LG = G(\mathcal{K})$ were an algebraic group these would be the actual tangent and cotangent bundles over $\mathcal{G}$ and Λ_λ would be the conormal bundle $T^*_{\mathcal{G}_\lambda}\mathcal{G}$ (we identify the dual of the ind-pro vector space $\mathfrak{g}_\mathcal{K}$ with $\mathfrak{g}_\mathcal{K}$ by a bilinear form, $B(x,y) = Res_{t=0}\ \chi(x,y)dt,\ x,y \in \mathfrak{g}_\mathcal{K}$, where χ is the $\mathcal{K}$-linear extension of an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$). Moreover, since we are only interested in finitely many P-orbits at a time, we should be able to pass to a finite dimensional situation by taking quotients with respect to some congruence subgroup of P. Though $\mathcal{G}$ is not smooth a precise version of these remarks is formulated in the next lemma which is explained in (2.3) and (2.4).

2.2. Lemma [KT,GG]. *For any finite closed union X of P-orbits in $\mathcal{G}$ there is a canonical projective system of embeddings $X \hookrightarrow \mathcal{U}(n),\ n >> 0,$ into finite dimensional manifolds $\mathcal{U}(n)$ such that at any point $L = g{\cdot}L_0$ in X, $T^*_L\mathcal{U}(n) = {}^g\mathfrak{p}/t^n\mathfrak{p}$ and $T^*_{\mathcal{G}_\nu}(\mathcal{U}(n))_L = {}^g\mathfrak{p} \cap \mathfrak{p}/t^n\mathfrak{p}$ if $L \in \mathcal{G}_\nu \subseteq X$.*

2.3. Embedding into an infinite dimensional manifold (see [Ka2,KT]). Consider formal Laurent series in t^{-1}, $\mathcal{K}_- \stackrel{\text{def}}{=} \mathbb{C}((t^{-1}))$ and the subring of polynomials $\mathcal{A} \stackrel{\text{def}}{=} \mathbb{C}[t]$. Now $\mathcal{G}$ embeds by $\mathcal{G} = G(\mathcal{K})/G(\mathcal{O}) \cong G([t,t^{-1}])/G(\mathcal{A}) \to G(\mathcal{K}_-)/G(\mathcal{A}) \stackrel{\text{def}}{=} \hat{\mathcal{G}}$ into the "completed loop Grassmannian" $\hat{\mathcal{G}}$. The congruence subgroups $U_k = G(t^{-k}\mathbb{C}[[t^{-1}]]) \stackrel{\text{def}}{=} \text{Ker}[G(\mathbb{C}[[t^{-1}]]) \to G(\mathbb{C}[[t^{-1}]]/t^k),\ k \geq 1,$ are pro-algebraic unipotent groups isomorphic to their Lie algebras $\mathfrak{u}_k$. Now $\hat{\mathcal{G}}$ is a separated scheme with an open cover $\mathcal{U}_\lambda = \lambda(z)\cdot\mathcal{U}_0$, $\lambda \in X_*(T)$, where $\mathcal{U}_0 \stackrel{\text{def}}{=} U_1\cdot L_0$ is isomorphic to U_1. So $\hat{\mathcal{G}}$ is a (pro)algebraic manifold modelled on $\mathbb{A}^\infty$, the product of a countable number of copies of $\mathbb{A}^1$. Each $\overline{\mathcal{G}_\mu}$ is a closed algebraic subvariety.

The tangent space at $L_0 \in \hat{\mathcal{G}}$ can be identified with $\mathfrak{u}_1 \cong \mathfrak{g}_{\mathcal{K}_-}/\mathfrak{g}_{\mathcal{A}}$. We use a bilinear form on $\mathfrak{g}_{\mathcal{K}_-}$, $B(x,y) = -Res_{t=\infty}\ \chi(x,y)dt$, where χ is the $\mathcal{K}_-$-linear extension of an invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$. It makes $\mathfrak{g}_{\mathcal{K}_-}$ self-dual and identifies $\mathfrak{g}_{\mathcal{A}}$ with its own annihilator so that $T^*_{L_0}\hat{\mathcal{G}} \cong \mathfrak{g}_{\mathcal{A}}$. In this way the bundle $T^*\hat{\mathcal{G}}$ is identified with the "$\mathfrak{g}_{\mathcal{K}_-}$-stabilizer bundle over $\hat{\mathcal{G}}$", i.e., $T^*_{g\cdot L_0}(\hat{\mathcal{G}}) = {}^g\mathfrak{g}_{\mathcal{A}}$, $g \in G(\mathcal{K}_-)$.

Now $\underset{\mathcal{G}_\lambda}{\cup} T^*_{\mathcal{G}_\lambda}\hat{\mathcal{G}}$ is the "$\mathfrak{g}_{\mathcal{A}}$-stabilizer scheme over $\mathcal{G}$", i.e., $T^*_{\mathcal{G}_\lambda}(\hat{\mathcal{G}})_{g\cdot L_0} = {}^g\mathfrak{g}_{\mathcal{A}} \cap \mathfrak{g}_{\mathcal{A}}$ for $g \in G(\mathbb{C}[t,t^{-1}])$ and $g\cdot L_0 \in \mathcal{G}_\lambda$. Since $P = G(\mathcal{O})$ does not act on $\hat{\mathcal{G}}$ one needs to check this formula for each $T^*_{\mathcal{G}_\lambda}\hat{\mathcal{G}}$ near L_λ. The formula follows from the isomorphism ${}^{\lambda(z)}U_1 \cap P \times {}^{\lambda(z)}U_1 \cap U_1 \to {}^{\lambda(z)}U_1 \to \mathcal{U}_\lambda$ which restricts to ${}^{\lambda(z)}U_1 \cap P \to \mathcal{U}_\lambda \cap \mathcal{G}_\lambda$.

2.4. Finite dimensional embeddings. Recall the finite union of orbits $X \subseteq \mathcal{G}$ and fix a neighborhood $\mathcal{U} \stackrel{\text{def}}{=} \underset{L_\lambda \in X}{\cup} \mathcal{U}_\lambda$ of X in $\hat{\mathcal{G}}$. For $n >> 0$, the congruence subgroup U_{n+1} acts locally freely on $\mathcal{U}$, $\mathcal{U}(n) \stackrel{\text{def}}{=} U_{n+1}\backslash\mathcal{U}$ is a finite dimensional manifold and the composition $X \subseteq \mathcal{U} \xrightarrow{p} U_{n+1}\backslash\mathcal{U} = \mathcal{U}(n)$, is an embedding.

For $L \in \mathcal{U}$, one has $T^*_{p(L)}\mathcal{U}(n) = T^*_L(\mathcal{U}) \cap t^n\mathfrak{u}_1$, since $(\mathfrak{u}_n)^\perp = (t^{1-n}\cdot\mathfrak{u}_1)^\perp = t^{n-1}\cdot(\mathfrak{u}_1)^\perp = t^{n-1}\cdot\mathfrak{u}_1$. So for $\mathcal{G}_\nu \subseteq X$, the fiber of the conormal $T^*_{\mathcal{G}_\nu}\mathcal{U}(n)$ at $L = g\cdot L_0 \in \mathcal{G}_\nu$ ($g \in G(\mathbb{C}[t,t^{-1}])$, equals ${}^g\mathfrak{g}_{\mathcal{A}} \cap \mathfrak{g}_{\mathcal{A}} \cap t^n\cdot\mathfrak{u}_1$. Now, since ${}^g\mathfrak{g}_{\mathcal{A}} \cap \mathfrak{g}_{\mathcal{A}}$ contains a complement $t^n\cdot\mathfrak{g}_{\mathcal{A}}$ to $t^n\cdot\mathfrak{u}_1$ in $\mathfrak{g}_{\mathcal{K}_-}$, we get identifications

$$T^*_{\mathcal{G}_\nu}(\mathcal{U}(n))_L = {}^g\mathfrak{g}_{\mathcal{A}} \cap \mathfrak{g}_{\mathcal{A}} \cap t^n\cdot\mathfrak{u}_1 \xrightarrow{\cong} ({}^g\mathfrak{g}_{\mathcal{A}} \cap \mathfrak{g}_{\mathcal{A}})/t^n\cdot\mathfrak{g}_{\mathcal{A}} \xrightarrow{\cong} ({}^g\mathfrak{p} \cap \mathfrak{p})/t^n\cdot\mathfrak{p}.$$

2.5. Theorem. *For different orbits $\mathcal{G}_\mu \neq \mathcal{G}_\lambda$, $e_{\mathcal{G}_\lambda,\mathcal{G}_\mu} = 0$.*

Proof. We can suppose that $\mathcal{G}_\lambda \subseteq \overline{\mathcal{G}_\mu}$. For $X = \overline{\mathcal{G}_\mu}$ we will use one of the embeddings $X \hookrightarrow \mathcal{U}(n)$ from lemma 2.2. Let P_n denote the n^{th} congruence subgroup $G(t^n\cdot\mathcal{O})$ of P. The group $Q_n \stackrel{\text{def}}{=} ({}^{\lambda(z)}P \cap P)/P_n$ acts on its Lie algebra $\mathfrak{q}_n \stackrel{\text{def}}{=} {}^{\lambda(z)}\mathfrak{p} \cap \mathfrak{p}/t^n\cdot\mathfrak{p}$ which lemma 2.2 identifies with the fiber at L_λ of $T^*_{\mathcal{G}_\lambda}\mathcal{U}(n)$. By proposition 1.5, the open subset of $\mathfrak{q}_n$ consisting of covectors generic for the stratification of X by P-orbits contains a covector ξ fixed by a maximal torus H of Q_n. Observe that the algebraic variety $\underset{\mathcal{G}_\nu \subseteq X}{\cup} T^*_{\mathcal{G}_\nu}\mathcal{U}(n)|X$ has an obvious P-action because of its description in lemma 2.2. Therefore the set of generic covectors is P-invariant and after Q_n-conjugation we can suppose that ξ is fixed by T.

Next, recall from the end of (2.3) that $({}^{\lambda(z)}U_1 \cap U_1)\cdot L_\lambda$ is a normal slice at L_λ to $\mathcal{G}_\lambda \cap \mathcal{U}_\lambda$ in $\mathcal{U}_\lambda$ and it is isomorphic to ${}^{\lambda(z)}U_1 \cap U_1 \cong {}^{\lambda(z)}\mathfrak{u}_1 \cap \mathfrak{u}_1$. Therefore, its image N in $\mathcal{U}(n)$ is a normal slice at L_λ to $\mathcal{G}_\lambda$ in $\mathcal{U}(n)$ isomorphic to ${}^{\lambda(z)}U_1 \cap U_1/U_n$ and then also to the vector space ${}^{\lambda(z)}\mathfrak{u}_1 \cap \mathfrak{u}_1/t^n\mathfrak{u}_1$. Moreover, the identification $({}^{\lambda(z)}\mathfrak{u}_1 \cap \mathfrak{u}_1/t^n\mathfrak{u}_1)^* \cong \mathfrak{q}_n$ given by the form B is precisely the identification $T^*_{\mathcal{G}_\lambda}(\mathcal{U}(n))_{L_\lambda} \cong \mathfrak{q}_n$ from (2.2) and it is T-equivariant. In this way ξ becomes a T-invariant linear function ϕ on the normal slice N with $d_{L_\lambda}\phi = \xi$. So for $t \in \mathbb{C}^*$, the hyperplane $\phi^{-1}(t) \subseteq N$ has a T action.

Finally, choose a small ball B as in 1.2 which is invariant under the compact form T_c of T. Then T_c acts on the complex link $\mathcal{L} = \mathcal{G}_\mu \cap \phi^{-1}(t) \cap B$. However $\mathcal{L}^T$ is empty since the T-fixed points in $\mathcal{G}$ are discrete, in fact $\mathcal{U}_\lambda^T = \{L_\lambda\}$. So $e_{\mathcal{G}_\lambda,\mathcal{G}_\mu} = \pm\chi_c(\mathcal{L})$ is zero by proposition 1.4. $\square$

2.6. Proof of the theorem 0.1. Due to the vanishing of Euler obstructions (theorem 2.5), the index theorem gives $c_{\mathcal{G}_\lambda} = (-1)^{\dim \mathcal{G}_\mu}\cdot\chi_{\mathcal{G}_\lambda}$. Now part (a) is obvious while (b) reduces to the claim $\chi_{\mathcal{G}_\mu}[IC(\overline{\mathcal{G}}_\lambda)] = (-1)^{\dim \mathcal{G}_\mu}\cdot m_\mu(L(\lambda))$ which was proved by Lusztig in [Lu2]. $\square$

2.7. The regular part of the Lagrangian. In the setting of lemma 2.2 we denote $\Lambda_{\lambda,n} \overset{\text{def}}{=} T^*_{\mathcal{G}_\lambda}\mathcal{U}(n)$ for $\mathcal{G}_\lambda \subseteq X$. The regular part of $\Lambda_{\lambda,n}$ is $\Lambda^r_{\lambda,n} \overset{\text{def}}{=} \Lambda_{\lambda,n} - \underset{\mathcal{G}_\lambda\neq\mathcal{G}_\nu\subseteq X}{\cup} \overline{\Lambda_{\nu,n}}$. We also define $\Lambda^{rs}_{\lambda,n}$ as the inverse of the regular semisimple part $\mathfrak{g}^{rs}$ of $\mathfrak{g}$ under the composition $\Lambda_{\lambda,n} \subseteq \mathfrak{p}/t^n\mathfrak{p} \times \mathcal{G} \to \mathfrak{p}/t^n\mathfrak{p} \to \mathfrak{g}$ (the last map is evaluation at 0).

2.8. Lemma. *Let $L \in \mathcal{G}$ and let $Q = P_L$ be the stabilizer of L in P. Denote by $\mathfrak{g}^{rs}$ the regular semisimple part of $\mathfrak{g}$ and let $\mathfrak{q}^{rs} \subseteq \mathfrak{q}$ be the inverse of $\mathfrak{g}^{rs}$ under the evaluation at 0 map $\mathfrak{q} \subseteq \mathfrak{p} \to \mathfrak{g}$.*

(a) $\mathfrak{q}^{rs} \subseteq \mathfrak{q}$ is open and dense.

(b) Any maximal torus H in Q is also maximal in $P$and its image $e(H) \subseteq G$ is maximal in G. If we use $\mathfrak{h} \cong e(\mathfrak{h}) \subseteq \mathfrak{g}$ to define the regular part $\mathfrak{h}^r$ of $\mathfrak{h}$, then $\mathfrak{q}^{rs}$ equals ${}^Q(\mathfrak{h}^r \oplus t\mathcal{O}\cdot\mathfrak{h})$.

(c) Any $x \in \mathfrak{q}^{rs}$ is centralized by precisely one maximal torus H in Q and $Z_\mathfrak{q}(x) = \mathcal{O}\cdot\mathfrak{h} = Z_\mathfrak{p}(x)$.

(d) Let $L \in \mathcal{G}_\lambda \subseteq X \subseteq \mathcal{U}(m)$ as in lemma 2.2 and suppose that λ is dominant. Let $x \in \mathfrak{q}^{rs}$ and $\mathfrak{h}_{x,n}$ be the image of $Z_\mathfrak{p}(x) = Z_\mathfrak{q}(x)$ in $\mathfrak{p}/t^n\cdot\mathfrak{p}$. For $n >> m \geq 0$, the commutative Lie algebra $\mathfrak{h}_{x,n}$ acts on $T^*_L\mathcal{U}(m)$ and the eigenfunctionals are in bijection with $\Delta \cup \{0\}$ (Δ = $\mathfrak{t}$-roots in $\mathfrak{g}$). The dimension of the generalized eigenspace corresponding to $\alpha \in \Delta \cup 0$ is $m- <\alpha,\lambda>$.

Proof. Using the P-action on $\mathcal{G}$ we can suppose that $L = L_\lambda$ for some $\lambda \in X_*(T)$. Now $Q = P \cap {}^{\lambda(z)}P = LV\cdot(Q \cap G(t\mathcal{O}))$ where $L\cdot V$ is a Levi decomposition of a parabolic subgroup of G, L is the centralizer of $\lambda(z)$ in G and $\mathfrak{v}$ is the sum of root spaces $\mathfrak{g}_\alpha$ for roots α with $<\alpha,\lambda>$ negative.

Now $\mathfrak{q} = (\mathfrak{l}\oplus\mathfrak{v})\oplus\mathfrak{q}\cap t\cdot\mathfrak{p}$ and $\mathfrak{q}^{rs} = (\mathfrak{l}\oplus\mathfrak{v})\cap\mathfrak{g}^{rs}\oplus\mathfrak{q}\cap t\cdot\mathfrak{p} = (\mathfrak{l}\cap\mathfrak{g}^{rs})\oplus(\mathfrak{v}\oplus\mathfrak{q}\cap t\cdot\mathfrak{p})$, so (a) is obvious. Up to Q-conjugation, we can choose H to be our fixed Cartan subgroup T, so $T(\mathcal{O}) \subseteq Q$ and $\mathfrak{t}_\mathcal{O} \subseteq \mathfrak{q}$. Since ${}^{LV}(\mathfrak{t}^r) = \mathfrak{l} \cap \mathfrak{g}^{rs} \oplus \mathfrak{v}$ and ${}^{Q\cap G(t\cdot\mathcal{O})}(\mathfrak{t}^r) \oplus t\cdot\mathfrak{t}_\mathcal{O} = \mathfrak{t}^r \oplus \mathfrak{q} \cap t\cdot\mathfrak{p}$, claim (b) follows.

In (c), by Q-conjugation we can suppose that x is in $\mathfrak{t}^r \oplus t\cdot\mathfrak{t}_\mathcal{O}$ and then the claim is clear. In (d) we can again suppose that $L = L_\lambda$ and $x \in \mathfrak{t}^r \oplus t\cdot\mathfrak{t}_\mathcal{O}$. Then $Z_\mathfrak{p}(x) = \mathfrak{t}_\mathcal{O}$ and $T^*_L\mathcal{U}(m)$ is by (2.2) the quotient of ${}^{\lambda(z)}\mathfrak{p} = \underset{\alpha\in\Delta\cup 0}{\oplus} {}^{\lambda(z)}(\mathcal{O} \otimes \mathfrak{g}_\alpha) = \underset{\alpha\in\Delta\cup 0}{\oplus} t^{<\alpha,\lambda>}\mathcal{O} \otimes \mathfrak{g}_\alpha$ by $t^m\cdot\mathfrak{p}$. Now $\mathfrak{h}_{x,n} = \mathfrak{t}_\mathcal{O}/t^n\mathfrak{t}_\mathcal{O}$ acts on $T^*_L\mathcal{U}(m)$ if $n+ <\alpha,\lambda> \geq m$, $\alpha \in \Delta \cup \{0\}$, so it certainly acts whenever $n >> m \geq 0$. Then $t^{<\alpha,\lambda>}\mathcal{O} \otimes \mathfrak{g}_\alpha/t^m\mathcal{O} \otimes \mathfrak{g}_\alpha$ is the generalized $\mathfrak{h}_{x,n}$-eigenspace for the eigenfunctional $\mathfrak{t}_\mathcal{O}/t^n\cdot\mathfrak{t}_\mathcal{O} \xrightarrow{t\mapsto 0} \mathfrak{t} \xrightarrow{\alpha} \mathbb{C}$. $\square$

2.9. Theorem. *If $\mathcal{G}_\lambda \subseteq X \subseteq \mathcal{U}(n)$ as in lemma 2.2, then $\Lambda^{rs}_{\lambda,n} \subseteq \Lambda^r_{\lambda,n}$ is open and dense.*

Proof. The idea is that the fibers of $\Lambda_n \stackrel{\text{def}}{=} \underset{\mathcal{G}_\mu \subseteq X}{\cup} \Lambda_{\mu,n} \to X$ are Lie algebras and we study Λ_n through its action on the vector bundles $T^*\mathcal{U}(m)|X$ for $n >> m \geq 0$ (it suffices to prove the theorem for one n!).

The theorem claims that it is impossible to have a sequence $(L_k, x_k) \in \Lambda_\lambda$ that converges to $(L, x) \in \Lambda^{rs}_\mu$ for $\mathcal{G}_\mu \subseteq \partial\mathcal{G}_\lambda$. If this happens, then by lemma (2.8.a) we can then suppose that $(L_k, x_k) \in \Lambda^{rs}$. Denote by $\mathfrak{h}_{y,n}$ the image of $Z_\mathfrak{p}(y)$ in $\mathfrak{p}/t^n{\cdot}\mathfrak{p}$. Then $\mathfrak{h}_{x_k,n}$ converges to $\mathfrak{h}_{x,n}$ in $Gr(\mathfrak{p}/t^n\mathfrak{p})$ since both have the same dimension $n{\cdot}\dim\mathfrak{t}$ by (2.8d). Now, for $n >> m \geq 0$, the action of $\mathfrak{h}_{x_k,n}$ on $T^*_{L_k}\mathcal{U}(m)$ (see 2.8d) converges to the action of $\mathfrak{h}_{x,n}$ on $T^*_L\mathcal{U}(m)$. Moreover, this is still true after we take quotients by the generalized eigenspaces corresponding to the zero eigenfunctionals since the dimension of these eigenspaces does not change by (2.8d). Therefore, the comparison of the dimensions of the largest generalized eigenspaces - see (2.8d) again - gives $n+ < \alpha_0, \mu > \geq \ n+ < \alpha_0, \lambda >$ for the B-highest root α_0. However, since $\mathcal{G}_\mu \subseteq \partial\mathcal{G}_\lambda$, $\lambda - \mu$ is a non-zero sum of positive roots and $< \alpha_0, \lambda - \mu >$ is strictly positive since α_0 is dominant.

2.10. Theorem. *The microlocal fundamental group correspoding to the stratum $\mathcal{G}_\lambda$ in $\mathcal{G}$ is a quotient of the "relative braid group" $B_{G,L} \stackrel{\text{def}}{=} \pi_1(\mathfrak{l} \cap \mathfrak{g}^{rs})$ corresponding to the Levi factor $L = Z_G(\lambda(z))$ of G.*

More precisely, whenever $\mathcal{G}_\lambda \subseteq X \subseteq \mathcal{U}(n)$ as in lemma 2.2, then $\pi_1(\Lambda^r_{\lambda,n})$ is a quotient of $\pi_1(\Lambda^{rs}_{\lambda,n}) \cong B_{G,L}$.

Proof. Since the complement of $\Lambda^{rs}_{\lambda,n}$ in $\Lambda^r_{\lambda,n}$ is a proper algebraic subvariety, the map $\pi_1(\Lambda^{rs}_{\lambda,n}) \to \pi_1(\Lambda^r_{\lambda,n})$ is surjective. Next, consider the projection $\Lambda^{rs}_{\lambda,n} \to \mathcal{G}_\lambda$ with the fiber at L_λ isomorphic to a product of $F = \mathfrak{l} \cap \mathfrak{g}^{rs}$ with a vector space (see the proof of the lemma 2.8). So it remains to show that the inclusion of the fiber at L_λ gives an isomorphism of the fundamental groups. Moreover, this will reduce to the same claim on the inclusion of the fiber $F \hookrightarrow G \underset{LV}{\times} F$ for a bundle over G/LV. The action of $\mathbb{C}^*$ on $\mathcal{K}$ by rotations $(s \circ f)(z) = f(c^{-1}{\cdot}z)$, $s \in \mathbb{C}^*$, produces an action of $\mathbb{C}^*$ on $\mathcal{G}_\lambda$ which contracts it to the G-orbit $G{\cdot}L_\lambda \subseteq \mathcal{G}_\lambda$. Therefore we can replace $\Lambda^{rs}_{\lambda,n}$ with its restriction to $G{\cdot}L_\lambda \cong G/LV$, and this restriction retracts to the subbundle $G \underset{LV}{\times} F$.

Recall that for a group G, one has a covering $\widetilde{\mathfrak{g}^{rs}}$ of $\mathfrak{g}^{rs}$ such that the fiber at $h \in \mathfrak{g}^{rs}$ consists of the Borel subalgebras of $\mathfrak{g}$ that contain h. The projection of $\widetilde{\mathfrak{g}^{rs}}$ to the flag variety G/B of G is the associated bundle $G \underset{B}{\times} \mathfrak{b} \cap \mathfrak{g}^{rs}$. Since $F \subseteq \mathfrak{l}^{rs}$, restriction of the similiar cover associated to L gives a covering $\tilde{F} = L \underset{B_L}{\times} \mathfrak{b}_L \cap \mathfrak{g}^{rs}$ of F. If we extend it to a covering $G \underset{LV}{\times} \tilde{F}$ of $G \underset{LV}{\times} F$, we can again replace $F \hookrightarrow G \underset{LV}{\times} F$ with $\tilde{F} \hookrightarrow G \underset{LV}{\times} \tilde{F}$, as one sees from the map of the long exact sequence of homotopy groups associated to the inclusion of coverings. For any Borel subgroup B_L of L, $B_L{\cdot}V$ is a Borel subgroup of G. The inclusion of $\tilde{F} \cong L \underset{B_L}{\times} \mathfrak{b}_L \cap \mathfrak{g}^{rs}$ into $G \underset{LV}{\times} \tilde{F} \cong G \underset{B_L \cdot V}{\times} \mathfrak{b}_L \cap \mathfrak{g}^{rs}$ gives an isomorphism of fundamental groups since the same is true for the inclusions of $\mathfrak{b}_L \cap \mathfrak{g}^{rs}$ into both associated bundles. To prove the last claim we pass to homotopic object by replacing $\mathfrak{b}_L \cap \mathfrak{g}^{rs}$ with $\mathfrak{b}_L \cap \mathfrak{g}^{rs}/[\mathfrak{b}_L, \mathfrak{b}_L]$ and observe that the new associated bundles are trivial since say $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ is a trivial B-module. □

2.11. Remark. One calls $B_G \overset{\text{def}}{=} \pi_1(\mathfrak{g}^{rs})$ the braid group and $B_G^0 \overset{\text{def}}{=} \pi_1(\mathfrak{t}^r)$ the pure braid group of G. In general $B_{G,L} \overset{\text{def}}{=} \pi_1(\mathfrak{l}\cap\mathfrak{g}^{rs})$ is isomorphic to $\pi_1(W(L)\backslash\mathfrak{t}^r)$ and one has an extension $0 \to B_G^0 \to B_{G,L} \to W(L) \to 0$.

3. Nilpotent orbits

In this section, we reduce the computation of Euler obstructions for nilpotent orbits to the case when the smaller orbit is distinguished and we derive a number of consequences.

3.1. Centralizers of nilpotent elements. Let $\mathcal{N}$ be the nilpotent cone in the Lie algebra $\mathfrak{g}$ of a reductive group G. For $e \in \mathcal{N}$, choose a $\mathfrak{sl}_2$ triple (e,h,f) and let $\mathfrak{s}$ be the corresponding three dimensional subalgebra in $\mathfrak{g}$. Let $Z_G(\mathfrak{s})$ and $Z_{\mathfrak{g}}(\mathfrak{s})$ be the centralizers of $\mathfrak{s}$ in G and $\mathfrak{g}$ respectively. $Z_G(\mathfrak{s})$ is a maximal reductive subgroup in $Z_G(e)$. It follows that a Cartan subgroup C of $Z_G(\mathfrak{s})$ is also a Cartan subgroup of $Z_G(e)$. Finally, our choice of C gives a Levi factor $M \overset{\text{def}}{=} Z_G(C)$ of G.

A nilpotent element e is called distinguished if the only tori $A \subseteq G$ that centralize e are in the center of G. This is equivalent to (i) C is the connected component of the center of G, and to (ii) $M = G$. Note that e is always a distinguished element of $\mathfrak{m} = Lie(M)$. Indeed, if a torus A in M centralizes e, then A is a torus in $Z_G(e)$ commuting with the Cartan subgroup C, so $A \subseteq C \subseteq Z(M)$. The regular nilpotent element is always distinguished. For $\mathfrak{sl}(n)$, the only distinguished orbit is the regular nilpotent orbit but for classical algebras $so(n)$ or $sp(2n)$ distinguished orbits are characterized by having distinct partitions in their Jordan forms.

3.2. Theorem. *Assume $G\cdot e$ is contained in the closure of the nilpotent orbit β. Then $e_{G\cdot e,\beta} = e_{M\cdot e,\beta\cap\mathfrak{m}}$, so that it suffices to compute Euler obstructions when e is distinguished.*

Proof. We consider the normal slice to $G\cdot e$ in $\mathfrak{g}$, $S = e + Z_{\mathfrak{g}}(f)$. Then $T_e^*S = Z_{\mathfrak{g}}(f)^*$ can be identified with $Z_{\mathfrak{g}}(e)$. Indeed, the Killing form induces a map $Z_{\mathfrak{g}}(e) \to Z_{\mathfrak{g}}(f)^*$ and it follows from $\mathfrak{sl}_2$ theory that the map is an isomorphism. By proposition 1.5, we see that a generic covector is centralized by a maximal torus in $Z_G(e)$, so after a $Z_G(e)$-conjugation we can choose a generic covector $\xi \in Z_{\mathfrak{g}}(e)$ so that ξ is fixed by C.

Now, ξ defines a linear function ϕ on S, $\phi(u) = B(\xi,u)$, $u \in S$. One has $\phi(e) = 0$ since e and ξ are commuting nilpotent elements and the differential $d_e\phi = \xi \in T_e^*(S) \cong T^*_{G\cdot e}(\mathfrak{g})_e$ is generic. Therefore, $e_{G\cdot e,\beta} = \chi_c\left(\beta\cap\phi^{-1}(t)\cap B\right)$ for a small ball B around e and appropriate t. Observe that C preserves e, S and ξ. The compact form T of C acts on the complex link $L \overset{\text{def}}{=} \beta\cap\phi^{-1}(t)\cap B$ provided the ball B is obtained from a T-invariant inner product on $\mathfrak{g}$. Now the fixed point set L^T gives the space which computes $e_{M\cdot e,\beta\cap\mathfrak{m}}$ and hence $\chi_c(L) = \chi_c(L^T)$.

3.3. Lemma. *Let $\alpha \subseteq \partial\beta$ be nilpotent orbits. In the following cases, $\beta\cap\mathfrak{m}$ is empty, so the Euler obstruction $e_{\alpha,\beta} = 0$.*
(a) $\mathfrak{g} = \mathfrak{sl}(n)$.
(b) $\alpha = \{0\}$.
(c) β is distinguished and α is not.

Proof. Let $e \in \alpha$. For (c), a Cartan C in $Z_G(e)_0$ is a nontrivial torus, so it fixes no point in β, i.e., β does not meet $\mathfrak{m} = Z_{\mathfrak{g}}(C)$. For (b), the centralizer of $e = 0$ is G

and $\mathfrak{m} = \mathfrak{t}$ does not meet β. For (a), $\mathfrak{m}$ is a centralizer of a torus, hence modulo the center it is a product of $\mathfrak{sl}(k)$'s. Now, since e is distinguished in $\mathfrak{m}$, it is a regular nilpotent in $\mathfrak{m}$. $\square$

Remark. Part (a) is actually equivalent to the vanishing of the Euler obstruction for the loop Grassmannians of type A. The reason is the equivalence between local geometry of P-orbits in the loop Grassmannians of type A and the geometry of nilpotent orbits for type A (this is an extension of Lusztig's embedding of the nilpotent cone into the Grassmannian from [Lu1]). $\square$

3.4. Corollary. *Let $\mathfrak{g} = \mathfrak{sl}(n)$ and let α and β be distinct nilpotent orbits and $i : \alpha \hookrightarrow \mathfrak{g}$. Then*

(a) $CC(i_!(\mathbb{C}_\beta)) = \overline{T^*_\beta(\mathfrak{g})}$.

(c) The microlocal multiplicity $c_\alpha(IC(\beta))$ of the intesection homology sheaf of $\bar\beta$ along the orbit α, is the multiplicity of the Springer representation $V(\beta)$ of the permutation group S_n in the cohomology $H^*(\mathcal{B}_e)$ of the Springer fiber at $e \in \alpha$ ($\mathcal{B}$ is the flag variety of $\mathfrak{g}$ and $\mathcal{B}_e$ is the zero set of the vector field given by e).

Proof. The statements follow from (3.3a) and the index formula since the stalk of $IC(\beta)$ at α equals $\mathrm{Hom}_{S_n}[V(\beta), H^*(\mathcal{B}_e)]$ (see [Lu1],[BM]).

3.5. Corollary (cf. [HK, 5.4.1]). *For any reductive algebra $\mathfrak{g}$, the multiplicity of $T^*_0(\mathfrak{g}) = 0 \times \mathfrak{g}^*$ in $CC(IC(\beta))$ is the dimension of the Springer representation $V(\beta)$.*

Proof. The Euler obstruction statement (3.3b) implies that the multiplicity of $0 \times \mathfrak{g}^*$ is the dimension of the stalk at 0. The isomorphism $IC(\beta)_0 \cong V(\beta)$ follows from [BM].

3.6. Subregular singularities.

In the remainder of the section we compute the Euler obstruction for the subregular nilpotent orbit $\mathcal{O}_{sr}$ and the regular nilpotent orbit $\mathcal{O}_r$ for a simple Lie algebra $\mathfrak{g}$. The Euler obstruction is either 0 or -1, depending on whether a certain variety is rational. We hope to relate this observation with results of Bernstein-Kazhdan and Hales [BK, Ha] in the future.

3.7. Theorem. *$e_{\mathcal{O}_{sr},\mathcal{O}_r} = 0$ if $\mathfrak{g} = A_n$ or B_n and $e_{\mathcal{O}_{sr},\mathcal{O}_r} = 1$ otherwise.*

3.8. Remark. In other words, in this case the Euler obstruction is non-trivial precisely if the smaller orbit is distinguished. The result may be known, at least in an equivalent form. We include a proof for completeness and to emphasize the appearance of nonrational varieties.

Proof. Let $e \in \mathcal{O}_{sr}$ and let $S = e + Z_{\mathfrak{g}}(f)$ be the normal slice. Then the germ $(e, S \cap \mathcal{N})$ is analytically isomorphic to the germ $(\{0\}, X)$, where X is one of the following surfaces (see [Sl], e.g.).

(A_n) $X = \{(x,y,z) | x^{n+1} + yz = 0\}, n \geq 1$

(D_n) $X = \{(x,y,z) | x^{n-1} + xy^2 + z^2 = 0\}, n \geq 4$

(E_6) $X = \{(x,y,z) | x^4 + y^3 + z^2 = 0\}$

(E_7) $X = \{(x,y,z) | x^3 y + y^3 + z^2 = 0\}$

(E_8) $X = \{(x,y,z) | x^5 + y^3 + z^2 = 0\}$

The labels given for the surfaces correspond to the subregular singularity for the corresponding simple Lie algebra and the cases of B_n, C_n, G_2, and F_4 coincide respectively with A_{2n-1}, D_{n+1}, D_4, and E_6. The subregular orbit is not distinguished precisely in the A_n and B_n cases. So these cases follow from lemma 3.3c. Moreover, the curve $\phi^{-1}(t)$ is rational since it carries a nontrivial action of $C \cong \mathbb{C}^*$. It remains to consider the surfaces D_n and E_n.

The main difficulty is to identify a generic function ϕ. However, Teissier has shown that for a hypersurface X with isolated singularity at 0 in affine space, a linear function ϕ is generic if its orthogonal hyperplane H minimizes the Milnor number $\mu(X \cap H, 0)$ among all hyperplanes [Te, chap. 2, Rem. 1.6] or [Lé, 1.2.3]. Recall that the Milnor number for an analytic hypersurface germ $Y = Z(f)$ in $\mathbb{C}^m$ with isolated singularity at 0 is $\mu(Y, 0) = \dim\ \mathbb{C}[z_1, \dots, z_m]/(f_{z_1}, \dots, f_{z_n})$ (f_{z_i} is the partial derivative of f with respect to z_i).

It is routine to check that we may take $\phi(x, y, z) = x$ for E_n, and $\phi(x, y, z) = x+y$ for D_n. In the case of E_n, the affine curve $\phi^{-1}(t) \cap X$ is of the form $y^3 + z^2 = \varepsilon$ for E_6 and E_8 and $y^3 + z^2 = \varepsilon y$ for E_7 (for some small ε). In any case we get a smooth elliptic curve with a point missing. After intersecting with a small ball, instead of a point we are removing a contractible closed ball. So $\chi_c = -1$.

The case of D_n is only slightly more complicated. The affine curve we obtain is hyperelliptic, but almost all topology is concentrated near the singular point at infinity. After intersecting with a small ball we get topologically the same space as in E_n-cases.

4. Character sheaves

In this section we consider microlocal aspects of Lusztig's theory of character sheaves on a reductive complex algebraic group G from [Lu3,Lu4]. In [MV1], various equivalent definitions are given. In particular, character sheaves are precisely the irreducible G-equivariant perverse sheaves $\mathcal{L}$ on G with nilpotent characteristic variety, i.e., $Ch(\mathcal{L}) \subseteq T^*G$ lies in $G \times \mathcal{N} \subseteq G \times \mathfrak{g} \cong G \times \mathfrak{g}^* \cong T^*G$. Here we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by a fixed nondegenerate invariant symmetric form on $\mathfrak{g}$. Similiarly, we will often identify the unipotent cone $\mathcal{U} \subseteq G$ with the nilpotent cone $\mathcal{N} \subseteq \mathfrak{g}$ by the exponential map. This section has some intersection with work of Ginzburg, especially [Gi3].

4.1. Lusztig stratification. Character sheaves are known to be constructible with respect to Lusztig's stratification of G. The stratum through an element with a Jordan decomposition $g = s{\cdot}u$ consists of all G-orbits through $u{\cdot}Z_r(L)$ where $L = Z_G(s)_0$ is the connected part of the centralizer of s and the regular part of the center of L is $Z_r(L) = \{z \in Z(L),\ Z_G(z) = L\}$. To the stratum S we associate the unipotent orbit through u and the corresponding nilpotent orbit.

We will call such a stratum S distinguished if u is a distinguished unipotent element of $Z_G(s)_0$. For instance all Lusztig strata S lying in the regular set G^r are distinguished since for $g = us \in S$, u is regular in $Z_G(s)_0$. For $G = GL_n$ these are all the distinguished strata - if u is distinguished in $Z_G(s) \cong \prod GL(n_i)$, then u is regular in $Z_G(s)$.

4.2. Lemma. *(a) For a character sheaf $\mathcal{L}$ on G, and a Lusztig stratum $S \subseteq G$, $c_S(\mathcal{L}) = 0$, unless S is distinguished. (b) For any distinguished stratum $S \subseteq G$, the closure of the image of $T^*_S G$ under the moment map $T^*G \to \mathfrak{g}^* \cong \mathfrak{g}$ is the closure of the nilpotent orbit associated to the Lusztig stratum S.*

Proof. Let $g = us \in S$, $L = Z_G(s)_0$ and let $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$ be the semisimple part of its Lie algebra. $T^*_S(G)_g$ is the part of $T^*_g(G) \cong \mathfrak{g}$ orthogonal to $T_g(S) = T_g\,(Z(L)) +$

$T_g({}^G g) = Z(\mathfrak{l}) + (Ad\ g - 1)\mathfrak{g}$, i.e.,

$$T^*_S(G)_g = Z(\mathfrak{l})^\perp \cap Z_\mathfrak{g}(g) = Z(\mathfrak{l})^\perp \cap Z_\mathfrak{l}(u) = Z_\mathfrak{s}(u).$$

It lies in the nilpotent cone iff u is distinguished in L, and this is the claim (a). Part (b) reduces to the claim that $Z_\mathfrak{s}(u) \cap \mathcal{O}$ is dense in $Z_\mathfrak{s}(u)$ found in the next lemma. $\square$

4.3. Lemma. *For a semisimple Lie algebra $\mathfrak{g}$ we define an involution on the Lagrangian subvarieties of $T^*\mathfrak{g}$ by $\mathcal{F}(x,y) = (y,-x)$ (we identify $T^*\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ with $\mathfrak{g} \oplus \mathfrak{g}$).*

For a nilpotent orbit $\mathcal{O} \subseteq \mathcal{N}$, $\mathcal{F}(\overline{T^*_\mathcal{O}\mathfrak{g}})$ lies above the nilpotent cone $\mathcal{N}$ iff $\mathcal{O}$ is distinguished. If $\mathcal{O}$ is distinguished then $\mathcal{F}(\overline{T^*_\mathcal{O}\mathfrak{g}}) = \overline{T^*_\mathcal{O}\mathfrak{g}}$. In particular, for an element e of a distinguished nilpotent orbit $\mathcal{O}$, $Z_\mathfrak{g}(e) \cap \mathcal{O}$ is dense in $Z_\mathfrak{g}(e)$.

Proof. Since $T^*_\mathcal{O}(\mathfrak{g})$ consists of all pairs (e,x) with $e \in \mathcal{O}$ and $x \in Z_\mathfrak{g}(e)$, $pr_1(\mathcal{F}\,\overline{T^*_\mathcal{O}\mathfrak{g}})$ lies in $\mathcal{N}$ if and only if $Z_\mathfrak{g}(e) \subseteq \mathcal{N}$, i.e., precisely when $\mathcal{O}$ is distinguished.

Then $\overline{T^*_\mathcal{O}\mathfrak{g}} = \overline{T^*_{\mathcal{O}'}\mathfrak{g}}$ for some nilpotent orbit $\mathcal{O}'$. Indeed, $\mathcal{F}\left(\overline{T^*_\mathcal{O}\mathfrak{g}}\right)$ is an irreducible Lagrangian subvariety of $T^*\mathfrak{g}$, hence of the form $\overline{T^*_\alpha\mathfrak{g}}$ for some connected submanifold $\alpha \subseteq \mathfrak{g}$. Since $\alpha \subseteq \mathcal{N}$ and $\overline{T^*_\mathcal{O}\mathfrak{g}}$ is G-invariant, α contains a dense orbit $\mathcal{O}'$ and then $\overline{T^*_\alpha\mathfrak{g}} = \overline{T^*_{\mathcal{O}'}\mathfrak{g}}$.

Now, since for $e \in \mathcal{O}$, e lies in $Z_\mathfrak{g}(e)$, so we have $\mathcal{O} \subseteq pr_2\overline{T^*_\mathcal{O}\mathfrak{g}} = pr_1\overline{T^*_{\mathcal{O}'}\mathfrak{g}} = \overline{\mathcal{O}'}$. By symmetry we have $\mathcal{O}' = \mathcal{O}$.

4.4. Character sheaves on $\mathfrak{g}$. We define these as the irreducible summands $\mathcal{L}^\mathfrak{g}_\tau$ of the induced sheaf $Ind^\mathfrak{g}_\mathfrak{t}\ \mathbb{C}_\mathfrak{t}[\dim\mathfrak{t}] = \underset{\tau}{\oplus}\tau \otimes \mathcal{L}^\mathfrak{g}_\tau$, indexed by the irreducible representations τ of the Weyl group W of the pair $T \subseteq G$ according to the geometric Springer construction ([BM]). These are constructible with respect to the Lusztig stratification on $\mathfrak{g}$ (the stratum through $x = s + n$ is ${}^G(n + Z_r[Z_\mathfrak{g}(s)])$.

We will denote by $\hat{W}$ the set of irreducible representations of W and by $sgn \in \hat{W}$ the sign representation. As in [BM], we associate to a nilpotent orbit $\mathcal{O}$ irreducible representations $V(\mathcal{O})$ of W. Let $\mathcal{F}_\mathfrak{g}$ be the Fourier transform on perverse sheaves on $\mathfrak{g}$, then

(4.4a) $\mathcal{L}^\mathfrak{g}_{V(\mathcal{O})}|\mathcal{N} = IC(\overline{\mathcal{O}})[\dim\mathfrak{t}]$ (see [BM]),

(4.4b) $\mathcal{F}_\mathfrak{g}\ IC(\overline{\mathcal{O}}) \cong \mathcal{L}^\mathfrak{g}_{V(\mathcal{O})\otimes sgn}$ ([HK, 5.2).

(4.4c) If $\mathcal{O}$ is a nilpotent orbit associated to a Lusztig stratum $S \subseteq \mathfrak{g}$ then $\chi_S = \chi_\mathcal{O}$ on each of the sheaves $\mathcal{L}^\mathfrak{g}_\tau$ ([BM]).

(4.4d) The stalk of $IC(\overline{\mathcal{O}})$ at the orbit $\mathcal{O}_L$ that contains a regular nilpotent element in a Levi factor $\mathfrak{l}$ is isomorphic to $\mathrm{Hom}_W[V(\mathcal{O}), Ind^W_{W(L)}\mathbb{C}]$ ([BM]).

(4.4e) If $S \subseteq \mathfrak{g}$ is a distinguished stratum, (i.e., the associated nilpotent orbit $\mathcal{O}$ is distinguished), then $\mathcal{F}\ \overline{T^*_S\mathfrak{g}} = \overline{T^*_\mathcal{O}\mathfrak{g}}$ (the left hand side is a Lagrangian in $T^*\mathfrak{g}$ and the analogue of (4.2b) for Lie algebras shows that its projection to $\mathfrak{g}$ is $\overline{\mathcal{O}}$).

4.5. The case $G = GL(n)$. Let us recall the parametrization from [Lu4] of character sheaves on G by conjugacy classes of elements $\check{g} = \check{u}\check{s}$ of the dual group $\check{G} \cong GL(n)$. If we fix dual Cartan subgroups $T \subseteq G$ and $\check{T} \subseteq \check{G}$ so that the semisimple part $\check{s}$ of $\check{g}$ lies in $\check{T}$, then $\check{s}$ defines a character sheaf $\mathcal{L}^T_{\check{s}}$ on T - the local system with monodromy $\check{s}$. Moreover, $\check{s}$ determines a Levi factor $\check{L} = Z_{\check{G}}s \subseteq \check{G}$ and therefore a Levi factor $L \supseteq T$ in G. Since the Weyl group $W(L)$ of L with respect to

T stabilizes $\mathcal{L}_{\check{s}}$, it acts on the induced character sheaf $Ind_T^L\,\mathcal{L}_{\check{s}}^T$ on L as in the Borho-MacPherson version of the Springer construction. So we have a decomposition according to the irreducible representations π of $W(L)$: $Ind_T^L\,\mathcal{L}_{\check{s}}^T = \oplus \pi\otimes\mathcal{L}_{\check{s},\pi}^L$, with the irreducible constituents $\mathcal{L}_{\check{s},\pi}^L \stackrel{\text{def}}{=} Hom_{W(L)}(\pi, Ind_T^L\,\mathcal{L}_{\check{s}}^T)$. Finally, the character sheaf on G associated to $\check{g}$ is $\mathcal{L}_{\check{g}}^G \stackrel{\text{def}}{=} Ind_L^G\,\mathcal{L}_{\check{s},\pi(\check{u})}^L$, where $\pi(\check{u})$ is the representation of $W(\check{L}) = W(L)$ associated to the unipotent orbit ${}^{\check{L}}\check{u}\subseteq\check{L}$ by the geometric Springer correspondence.

Regular Lusztig strata $S_\lambda\subseteq G^r$ and $S_\lambda^{\mathfrak{g}}\subseteq\mathfrak{g}^r$ are both parametrized by the partitions λ of n. Strata S_λ and $S_\lambda^{\mathfrak{g}}$ both consist of regular elements with Jordan blocks of size λ and the associated nilpotent orbit $\mathcal{O}_\lambda$ consists of nilpotent operators with Jordan blocks of size λ.

4.6. Theorem. *Let $G = GL(n)$. Denote by $\mathcal{D}$ the duality operation on character sheaves ([Lu4]).*

(a) For any character sheaf $\mathcal{L}$ on G and any distinguished stratum $S\subseteq G$ with the associated unipotent orbit $\mathcal{O}$

$$c_S(\mathcal{L}) = \chi_{\mathcal{O}}[\mathcal{D}(\mathcal{L})] = \chi_S[\mathcal{D}(\mathcal{L})].$$

(b) Let $\tau\in\hat{W}$, and let $\mathcal{O}$ be the nilpotent orbit such that $V(\mathcal{O})\cong\tau\otimes sgn$. Then

$$c_{S_\lambda^{\mathfrak{g}}}(\mathcal{L}_\tau^{\mathfrak{g}}) = \chi_{\mathcal{O}_\lambda}[IC(\overline{\mathcal{O}})] = (-1)^{\dim\mathfrak{t}}\chi_{\mathcal{O}_\lambda}(\mathcal{L}_{\tau\otimes sgn}^{\mathfrak{g}}) = (-1)^{\dim\mathfrak{t}}\chi_{S_\lambda^{\mathfrak{g}}}(\mathcal{L}_{\tau\otimes sgn}^{\mathfrak{g}})$$

and this is the dimension of the anti-invariants in the restriction of τ to the parabolic subgroup $W_\lambda\subseteq W = S_n$ of type λ.

(c) For $\check{g} = \check{s}\check{u}\in\check{G}$, the characteristic cycle of the character sheaf $\mathcal{L}_{\check{g}}^G$ is given by the maps between standard S_n-modules. If L is the Levi factor associated to $\check{s}$ as in (4.5), we have a sum over partitions λ of n

$$CC(\mathcal{L}_{\check{g}}^G) = \bigoplus_\lambda \dim\Big(Hom_W[Ind_{W(L)}^W\pi(\check{u}), Ind_{W_\lambda}^W sgn]\Big)\cdot\overline{T_{S_\lambda}^*G}.$$

Proof. For any G-equivariant perverse sheaf $\mathcal{A}$ on $\mathcal{N}$, the characteristic cycle $CC(\mathcal{F}_{\mathfrak{g}}\mathcal{A})$ of the Fourier transform of $\mathcal{A}$ equals $\mathcal{F}[CC(\mathcal{A})]$ for the operation $\mathcal{F}$ from (4.3) ([KaSc]). We know by (4.4e) that $\mathcal{F}(\overline{T_{S_\lambda^{\mathfrak{g}}}^*\mathfrak{g}}) = \overline{T_{\mathcal{O}_\lambda}^*\mathfrak{g}}$. So we get the first equality in

$$\text{(4.6.1)}\qquad c_{S_\lambda^{\mathfrak{g}}}[\mathcal{F}_{\mathfrak{g}}(\mathcal{A})] = c_{\mathcal{O}_\lambda}(\mathcal{A}) = \chi_{\mathcal{O}_\lambda}(\mathcal{A}),$$

and lemma 3.3a gives the second.

To prove (b) let $\mathcal{A} = IC(\overline{\mathcal{O}})$. Since $\mathcal{F}_{\mathfrak{g}}\mathcal{A}\cong\mathcal{L}_\tau^{\mathfrak{g}}$ by (4.4b), we get $c_{S_\lambda^{\mathfrak{g}}}(\mathcal{L}_\tau^{\mathfrak{g}}) = \chi_{\mathcal{O}_\lambda}[IC(\overline{\mathcal{O}})]$. Because of (4.4a) this is the same as $(-1)^{\dim\mathfrak{t}}\cdot\chi_{\mathcal{O}_\lambda}(\mathcal{L}_{\tau\otimes sgn}^{\mathfrak{g}})$ and by (4.4c) we can replace $\chi_{\mathcal{O}_\lambda}$ by $\chi_{S_\lambda^{\mathfrak{g}}}$. Finally, since the stalk at $\mathcal{O}_\lambda$ of $IC(\overline{\mathcal{O}})$ is concentrated in even degrees, by (4.4d), $\chi_{\mathcal{O}_\lambda}[IC(\overline{\mathcal{O}})]$ is the dimension of $\mathrm{Hom}_W(\tau\otimes sgn, Ind_{W_\lambda}^W\mathbb{C})\cong\mathrm{Hom}_{W_\lambda}[\tau|W_\lambda, sgn|W_\lambda]$, i.e., of the W_λ-anti-invariants in τ.

In order to prove (a), we will move everything from the group G to its Lie algebra $\mathfrak{g}$. So let us fix neighborhoods U of $1\in G$ and $U^{\mathfrak{g}}$ of $0\in\mathfrak{g}$ that are identified by the exponential map. Notice that this identifies Lusztig stratifications of $U\subseteq G$ and $U^{\mathfrak{g}}\subseteq\mathfrak{g}$ and in particular the strata $S_\lambda\cap U$ and $S_\lambda^{\mathfrak{g}}\cap U^{\mathfrak{g}}$.

Consider the specialization at 1 operation Sp_1 that takes perverse sheaves on G to the perverse sheaves on the tangent space $\mathfrak{g}$ at 1 ([KaSc]). We claim that for a character sheaf $\mathcal{L}$ on G and a pair of related strata $S \subseteq G$, $S^{\mathfrak{g}} \subseteq \mathfrak{g}$, one has $c_S(\mathcal{L}) = c_{S^{\mathfrak{g}}}(Sp_1\mathcal{L})$ and $\chi_S(\mathcal{L}) = \chi_{S^{\mathfrak{g}}}(Sp_1\mathcal{L})$.

Since all Lusztig strata in $GL(n)$ meet U, it suffices to show that

$$(4.6.2) \qquad (Sp_1\mathcal{L})|U^{\mathfrak{g}} \cong exp^*(\mathcal{L}|U).$$

The point is that both operations Sp_1 and exp^* are compatible with the induction and the Weyl group action, hence the problem reduces to the case when $G = T$ and the obvious claim $Sp_1(\mathcal{L}^T_{\check{s}}) \cong \mathbb{C}_{\mathfrak{t}}[\dim \mathfrak{t}]$. Another consequence of (4.6.2) is the explicit formula for the specialization of character sheaves $Sp_1(\mathcal{L}^G_{\check{g}}) \cong Ind^{\mathfrak{g}}_{\mathfrak{l}}\ \mathcal{L}^{\mathfrak{l}}_{\pi(\check{u})}$, which we will also write as a sum over irreducible representations τ of W, $\underset{\tau}{\oplus}\mathrm{Hom}_{W(L)}[\pi(\check{u}), \tau|W(L)] \otimes \mathcal{L}^{\mathfrak{g}}_{\tau}$ ([BM]).

Now it remains to show that

$$(4.6.3) \qquad Sp_1(\mathcal{D}\mathcal{L})\ |\mathcal{N} \cong \mathcal{F}_{\mathfrak{g}}(Sp_1\mathcal{L}),$$

since then for any distinguished stratum $S = S_\lambda$ in G,

$$c_S(\mathcal{L}) = c_{S^{\mathfrak{g}}_\lambda}(Sp_1\mathcal{L}) \overset{(4.6.1)}{=} \chi_{\mathcal{O}_\lambda}[\mathcal{F}_{\mathfrak{g}}Sp_1\mathcal{L}] \overset{(4.6.3)}{=}$$

$$\chi_{\mathcal{O}_\lambda}[Sp_1(\mathcal{D}\mathcal{L})|\mathcal{N}] = \chi_{\mathcal{O}_\lambda}[Sp_1(\mathcal{D}\mathcal{L})] = \chi_{\mathcal{O}}[\mathcal{D}\mathcal{L}].$$

Let $\mathcal{L} = \mathcal{L}_{\check{s}\check{u}}$ and recall that on the summands of induced sheaves the duality operation is given by $\mathcal{D}(\mathcal{L}_{\check{s}\check{u}}) = \mathcal{L}_{\check{s}\check{v}}$, where unipotent elements $\check{u}, \check{v}$ of $\check{L}$ lie in the opposite orbits, i.e., $\pi(\check{v}) = \pi(\check{u}) \otimes sgn_L$ for the sign representation sgn_L of $W(L)$ ([Lu4]). Now (4.6.3) follows from the explicit formula for $Sp_1\mathcal{L}_{\check{s}\check{u}}$ and the facts (4.4a) and (4.4b).

In (c), $c_{S_\lambda}(\mathcal{L}^G_{\check{g}}) = c_{S^{\mathfrak{g}}_\lambda}(Sp_1\mathcal{L}^G_{\check{g}})$ and

$$Sp_1\mathcal{L}^G_{\check{g}} = Ind^{\mathfrak{g}}_{\mathfrak{l}}\mathcal{L}^{\mathfrak{l}}_{\pi(\check{u})} \cong \underset{\tau\in\hat{W}}{\oplus} \mathrm{Hom}_{W(L)}[\pi(\check{u}), \tau] \otimes \mathcal{L}^{\mathfrak{g}}_{\tau},$$

hence by the part (b) $c_{S_\lambda}(\mathcal{L}^G_{\check{g}})$ is the dimension of

$$\underset{\tau\in\hat{W}}{\oplus} \mathrm{Hom}_{W(L)}[\pi(\check{u}), \tau] \otimes \mathrm{Hom}_{W_\lambda}[\tau, sgn|W_\lambda] \cong$$

$$\underset{\tau\in\hat{W}}{\oplus} \mathrm{Hom}_W[Ind^W_{W(L)}\pi(\check{u}), \tau] \otimes \mathrm{Hom}_W[\tau, Ind^W_{W_\lambda}sgn]$$

which is isomorphic to $\mathrm{Hom}_W[Ind^W_{W(L)}\pi(\check{u}), Ind^W_{W_\lambda}sgn]$.

4.7. Theorem. *Euler obstructions for the regular strata in the group GL_n (or in the Lie algebra gl_n) form the transition matrix for standard S_n-modules, i.e., in the Grothendieck group:*

$$[Ind^W_{W_\mu}sign] = (-1)^n \sum_\lambda e_{S_\mu,S_\lambda}\cdot[Ind^W_{W_\lambda}\mathbb{C}].$$

Proof. The Euler obstructions are the same for GL_n and gl_n, because near $0 \in \mathfrak{g}$ the exponential map identifies the Lusztig strata in G and $\mathfrak{g}$. So we will calculate in $\mathfrak{g}$ and we denote $S^{\mathfrak{g}}_\lambda$ simply by S_λ. Let β and β^{tr} be the nilpotent orbits such that $V(\beta^{tr}) \cong V(\beta) \otimes sgn$. We observe that

$$\dim \mathrm{Hom}_W[V(\beta), Ind^W_{W_\mu}\mathbb{C}] \overset{(4.4d)}{=} \chi_{\mathcal{O}_\mu}[IC(\overline{\beta})] \overset{(4.6.1)}{=}$$

$$c_{S_\mu}[\mathcal{F}_{\mathfrak{g}}\ IC(\overline{\beta})] \overset{(4.4b)}{=} c_{S_\mu}[\mathcal{L}^{\mathfrak{g}}_{V(\beta)\otimes sgn}],$$

Also,

$$\chi_{S_\lambda}[\mathcal{L}^{\mathfrak{g}}_{V(\beta)\otimes sgn}] \overset{(4.4c)}{=} \chi_{\mathcal{O}_\lambda}[\mathcal{L}^{\mathfrak{g}}_{V(\beta)\otimes sgn}] \overset{(4.4a)}{=} (-1)^{\dim \mathfrak{t}} \cdot \chi_{\mathcal{O}_\lambda}[IC(\overline{\beta^{tr}})] \overset{(4.4d)}{=}$$

$$(-1)^{\dim \mathfrak{t}} \cdot \dim \mathrm{Hom}_W[V(\beta)\otimes sgn, Ind^W_{W_\lambda}\mathbb{C}] = (-1)^{\dim \mathfrak{t}} \cdot \dim \mathrm{Hom}_W[V(\beta), Ind^W_{W_\lambda} sgn].$$

Now the index theorem gives $c_{S_\mu}(\mathcal{L}^{\mathfrak{g}}_{V(\beta)\otimes sgn}) = \sum_\lambda e_{S_\mu, S_\lambda} \cdot \chi_{S_\lambda}(\mathcal{L}^{\mathfrak{g}}_{V(\beta)\otimes sgn})$, so

$$\dim \mathrm{Hom}_W[V(\beta), Ind^W_{W_\mu}\mathbb{C}] = \sum_\lambda (-1)^{\dim \mathfrak{t}} e_{S_\mu, S_\lambda} \cdot \dim \mathrm{Hom}_W[V(\beta), Ind^W_{W_\lambda} sgn],$$

and this is the required statement. □

4.8. Configurations. Observe that $\mathfrak{t}/W$ is isomorphic to a normal slice to the smallest regular orbit in $\mathfrak{g}$, and for $G = GL_n$, the standard stratification of $\mathfrak{t}/W$ (stratifications of symmetric powers of a line by diagonals) is induced from the Lusztig stratification of the regular set in $\mathfrak{g}$. Since even the parity of the dimensions of the strata is the same, Euler obstructions are the same.

Now one can reproduce Laumon's computation of the characteristic cycle of the direct image of the constant sheaf along a partial symmetrization map $r_\mu : \mathfrak{t}/W_\mu \to \mathfrak{t}/W$ ([La], proof of 3.3.13) since the stalks of $(r_\mu)_* \mathbb{C}_{\mathfrak{t}/W_\mu}$ are obvious.

4.9. Cuspidal sheaves. The character sheaves supported on $\mathcal{U}$ coincide with the so-called cuspidal sheaves supported on $\mathcal{U}$, (see [Lu4], [Gi3], and also [Mi] for a simple proof in characteristic zero). These are the basic atoms of the theory of character sheaves.

4.10. Lemma. *Let G be a simple group. If $\mathcal{L}$ is a character sheaf supported on $\mathcal{U}$ then the support of $\mathcal{L}$ is the closure of the smallest distinguished unipotent orbit.*

Proof. This observation relies on Lusztig's case by case classification of cuspidal sheaves on the unipotent cone [Lu3] (so the claim is also true in any good characteristic). One can check the assertion for the classical groups from the description of distinguished orbits and for the exceptional groups by examining tables (see [Ca] for example). □

4.11. Theorem. *Let $\mathcal{L}$ be a cuspidal sheaf supported on a unipotent orbit β.*

(a) Its characteristic variety is irreducible: $CC(\mathcal{L}) = rk(\mathcal{L}) \cdot \overline{T^*_\beta(G)}$.

(b) The Euler characteristic of the stalk of $\mathcal{L}$ on any orbit $\alpha \subseteq \partial\beta$ is zero.

Part (a) is an unpublished result of Lusztig and (b) is a weak form of a mysterious result of Lusztig: $\mathcal{L}$ is clean, i.e., its stalks vanish on $\partial\beta$ [Lu4].

Proof. (a) Lemma 4.2 says that any orbit $\alpha \subseteq \bar{\beta}$ such that $T^*_\alpha\mathfrak{g}$ appears in $Ch(\mathcal{L})$ is distinguished. However, $\mathcal{L}$ is supported on a minimal distinguished orbit, hence $\alpha = \beta$.

(b) Again, since β is a minimal distinguished orbit, for any orbit $\alpha \subseteq \partial\beta$ one has $e_{\alpha,\beta} = 0$ (lemma 3.3c). Now $c_\alpha(\mathcal{L}) = \pm\chi_\alpha(\mathcal{L})$ by the index formula. However, the left side is zero by (a), hence the Euler characteristic of the stalk at α is zero.

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