ON THE WONDERFUL COMPACTIFICATION

SAM EVENS AND BENJAMIN JONES

Contents

1. Introduction

The purpose of these notes is to explain the construction of the wonderful compactification of a complex semisimple group of adjoint type. The wonderful compactification of a symmetric space was introduced by DeConcini and Procesi [5], and has been extensively studied in algebraic geometry. Of particular interest are the recent proofs of the Manin conjecture for the compactification in [16] and [12]. Intuitively, the wonderful compactification gives information about the group at infinity. In this connection, we mention the paper by He and Thomsen [14] showing that closures of different regular conjugacy classes coincide at infinity. The wonderful compactification has been used by Ginzburg in the study of character sheaves [11], and by Lusztig in his study of generalized character sheaves [15]. It is closely related to the Satake compactification and to various analytic compactifications [3]. It also plays an important role in Poisson geometry, as it is crucial for undersanding the geometry of a moduli space of Poisson homogeneous spaces [8, 9]. This list is not meant to be exhaustive, but to give some idea of how the wonderful compactification is related to the rest of mathematics. The reader may consult [18] and [19] for further references.

The appendix contains a construction of the wonderful compactification of a complex symmetric space due to DeConcini and Springer [7]. This construction does not specialize to the construction in Section 2 in the case when the symmetric space is a group, but it is equivalent, and the two constructions are conceptually similar.

These notes are based on lectures given by the first author at Hong Kong University of Science and Technology and by both authors at Notre Dame. They are largely based on work of DeConcini, Procesi, and Springer which is explained in the papers [5], [7], and the book [4]. The notes add little in content, and their purpose is to make the simple construction of the wonderful compactification more accessible to students and to mathematicians without an extensive background in algebraic groups and algebraic geometry.

We would like to thank Hong Kong University of Science and Technology for hospitality during preparation of a first draft of these notes, and would like to thank Jiang-Hua, Dragan Milicic, Allen Moy, Xuhua He, Francois Ledrappier, and Dennis Snow for useful comments and discussions.

1.1. **Notation.** In these notes, an algebraic variety is a complex quasi-projective variety, not necessarily irreducible. A subvariety is a locally closed subset of a variety.

2. Construction and basic properties of the wonderful compactification

We explain how to prove that the wondeful compactification of a semisimple complex group G of adjoint type is smooth, and describe its $G \times G$-orbit structure.

2.1. **Definition of the compactification.** Let G be a complex connected semisimple group with trivial center. Let $\tilde{G}$ be the simply connected cover of G, and choose a maximal torus $\tilde{T}$ contained in a Borel subgroup $\tilde{B}$ of $\tilde{G}$. We denote their images in G by $T \subset B$. We denote Lie algebras of algebraic groups by the corresponding gothic letter, so the Lie algebras of $T \subset B \subset G$, and of $\tilde{T} \subset \tilde{B} \subset \tilde{G}$, are $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$. Let $X^*(\tilde{T})$ be the group of characters of $\tilde{T}$. We write this group additively, so if $\lambda, \mu \in X^*(\tilde{T})$ we have by definition $(\lambda + \mu)(t) = \lambda(t)\mu(t)$ for all $t \in \tilde{T}$. Let $\Phi \subset X^*(\tilde{T})$ be the roots of $\tilde{T}$ in $\tilde{G}$, and take the positive roots Φ^+ to be the roots of $\tilde{T}$ in $\tilde{B}$. Let $\{\alpha_1, \dots, \alpha_l\}$ be the corresponding set of simple roots, where $l = \dim(\tilde{T})$. With these choices, for $\lambda, \mu \in X^*(\tilde{T})$, we say $\lambda \geq \mu$ if $\lambda - \mu = \sum_{\alpha \in \Phi^+} n_\alpha \alpha$ where the n_α are nonnegative integers. There is an embedding of the character group $X^*(T) \hookrightarrow X^*(\tilde{T})$ as the characters that are trivial on the center of G. If λ is in the image of this embedding, we may compute $\lambda(t)$ for $t \in T$.

Let $\tilde{B}^-$ be the opposite Borel of $\tilde{G}$ containing $\tilde{T}$ and let B^- be its image in G. Let U and U^- be the unipotent radicals of $\tilde{B}$ and $\tilde{B}^-$. The group homomorphism $\tilde{G} \to G$ restricts to an isomorphism on any unipotent subgroup, and we identify U and U^- with their images in G. If W is a representation of $\tilde{G}$, set W_μ for the μ-weight space for the $\tilde{T}$-action. If $v \in W_\mu$, then $U^- \cdot v \subset v + \sum_{\phi<\mu} W_\phi$.

Fix an irreducible representation $V = V(\lambda)$ of G with regular highest weight λ and choose a basis $v_0, \dots, v_n$ of T-weight vectors of V with the following properties:

(1) v_0 has weight λ;

(2) For $i \in \{1, \dots, l\}$, v_i has weight $\lambda - \alpha_i$;

(3) Let λ_i be the weight of v_i. Then if $\lambda_i < \lambda_j$, then $i > j$.

Remark 2.1. For $i = 0, \dots, l$, $\dim(V_{\lambda_i}) = 1$.

Remark 2.2. $U^- \cdot v_i - v_i$ is in the span of the v_j for $j > i$.

The induced action of $\tilde{G}$ on the projective space $\mathbb{P}(V)$ factors to give an action of G. Define $\mathbb{P}_0(V) = \{[\sum_{i=0}^n b_i v_i] : b_0 \neq 0\} \subset \mathbb{P}(V)$. The affine open set $\mathbb{P}_0(V) \cong \{v_0 + \sum_{i=1}^n b_i v_i\} \cong \mathbb{C}^n$. Further, $\mathbb{P}_0(V)$ is U^--stable by Remark 2.2 applied to v_0.

Also, the morphism $U^- \to U^- \cdot [v_0]$ is an isomorphism of algebraic varieties. Indeed, the stabilizer of $[v_0]$ in G is B, so the stabilizer in U^- of $[v_0]$ is trivial, and the result follows.

Lemma 2.3. *$U^- \mapsto U^- \cdot [v_0]$ is an isomorphism between U^- and the closed subvariety $U^- \cdot [v_0]$ of $\mathbb{P}_0(V)$.*

We have proved everything but the last statement, which follows by the following standard fact.

Lemma 2.4. *Let A be a unipotent algebraic group and let X be an affine A-variety. Then every A-orbit in X is closed.*

$\tilde{G}$ acts on V^* by the dual action, $g \cdot f = f \circ g^{-1}$. Choose a dual basis $v_0^*, \dots, v_n^*$ of V^*. Then each v_i^* is a weight vector for $\tilde{T}$ of weight $-\lambda_i$, and v_0^* is a highest weight vector with respect to the negative of the above choice of positive roots. Note that $\mathbb{P}_0(V) = \{[v] : v_0^*(v) \neq 0\}$.

Define $\mathbb{P}_0(V^*) = \{[\sum_{i=0}^n b_i v_i^*] : b_0 \neq 0\}$. Then U acts on $\mathbb{P}_0(V^*)$, and the morphism $U \mapsto U \cdot [v_0^*]$ is an isomorphism of varieties. We state the analogue of Lemma 2.3 for this action, which follows by reversing the choice of positive roots.

Lemma 2.5. *$U \mapsto U \cdot [v_0^*]$ is an isomorphism between U and the closed subvariety $U \cdot [v_0^*]$ of $\mathbb{P}_0(V^*)$.*

$\tilde{G} \times \tilde{G}$ acts on $\mathrm{End}(V)$ by the formula $(g_1, g_2) \cdot A = g_1 A g_2^{-1}$ and on $V \otimes V^*$ by linear extension of the formula $(g_1, g_2) \cdot v \otimes f = g_1 \cdot v \otimes g_2 \cdot f$. Then the canonical identification $V \otimes V^* \cong \mathrm{End}(V)$ is $\tilde{G} \times \tilde{G}$-equivariant, and we treat this identification as an equality. Then $\{v_i \otimes v_j^* : i, j = 0, \ldots n\}$ is a basis of $\mathrm{End}(V)$. As before, the $\tilde{G} \times \tilde{G}$-action on $\mathbb{P}(\mathrm{End}(V))$ descends to give a $G \times G$-action.

We consider the open set defined by

$$\mathbb{P}_0 = \{[\sum a_{ij} v_i \otimes v_j^*] : a_{00} \neq 0\} = \{[A] \in \mathbb{P}(\mathrm{End}(V)) : v_0^*(A \cdot v_0) \neq 0\}.$$

It follows from the above that $U^- T \times U$ preserves $\mathbb{P}_0$. Indeed, we use that fact that $U^- T$ preserves $\mathbb{P}_0(V)$ and U preserves $\mathbb{P}_0(V^*)$.

Define an embedding $\psi : G \to \mathbb{P}(\mathrm{End}(V))$ by $\psi(g) = [g]$, and note that ψ is $G \times G$-equivariant, where $G \times G$ acts on G by $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$. Let $X = \overline{\psi(G)}$.

2.2. **Geometry of the open affine piece.** Let $X_0 = X \cap \mathbb{P}_0$. Let the Weyl group $W = N_G(T)/T$, and for each $w \in W$, choose a representative $\dot{w} \in N_G(T)$.

Lemma 2.6. *$X_0 \cap \psi(G) = \psi(U^- T U)$.*

Proof. Clearly, $\psi(e) \in X_0$, and $\psi(U^- T U) = (U^- T \times U) \cdot \psi(e)$, so $\psi(U^- T U) \subset X_0$ since $\mathbb{P}_0$ is $U^- T \times U$ stable. This gives one inclusion.

For the other inclusion, recall that by the Bruhat decomposition, $G = \cup_{w \in W} U^- T w U$. Then $\dot{w} \cdot [v_0]$ is a weight vector of weight $w\lambda$. Since λ is regular, if $w \neq e$, then $\dot{w} \cdot [v_0] \notin X_0$. Thus, $\psi(\dot{w}) \notin X_0$. Now use $U^- T \times U$-stability of X_0 again.

Q.E.D.

Since $\psi(U^- T U)$ is dense in $\psi(G)$, it is dense in X, and it follows that X_0 is the closure $\overline{\psi(U^- T U)}$ in $\mathbb{P}_0$.

Define $Z = \overline{\psi(T)}$, the closure of $\psi(T)$ in $\mathbb{P}_0$.

Lemma 2.7. *$Z \cong \mathbb{C}^l$, where $l = \dim(T)$.*

Proof. Under the identification of $\mathrm{End}(V)$ with $V \otimes V^*$, the identity corresponds to $\sum_{i=0}^n v_i \otimes v_i^*$. Thus, for $\tilde{t} \in \tilde{T}$ projecting to $t \in T$,

$$\psi(t) = t \cdot \psi(e) = [\sum \tilde{t} \cdot v_k \otimes v_k^*] = [\sum \lambda_k(\tilde{t}) v_k \otimes v_k^*]$$

This is

$$[\lambda(\tilde{t}) v_0 \otimes v_0^* + \sum_{k>0} \lambda_k(\tilde{t}) v_k \otimes v_k^*] = [v_0 \otimes v_0^* + \sum_{k>0} \frac{\lambda_k(\tilde{t})}{\lambda(\tilde{t})} v_k \otimes v_k^*].$$

But

$$\frac{\lambda_k(\tilde{t})}{\lambda(\tilde{t})} = \frac{1}{\prod_{i=1}^l \alpha_i(t)^{n_{ik}}}$$

where $\lambda_k = \lambda - \sum_{i=1}^{l} n_{ik}\alpha_i$.

Thus,

$$\psi(t) = [v_0 \otimes v_0^* + \sum_{i=1}^{l} \frac{1}{\alpha_i(t)} v_i \otimes v_i^* + \sum_{k>l} \frac{1}{\prod_{i=1}^{l} \alpha_i(t)^{n_{ik}}} v_k \otimes v_k^*].$$

Define $F : \mathbb{C}^l \to \mathbb{P}_0$ by

$$F(z_1, \ldots, z_l) = [v_0 \otimes v_0^* + \sum_{i=1}^{l} z_i v_i \otimes v_i^* + \sum_{k>l} (\prod_{i=1}^{l} z_i^{n_{ik}}) v_k \otimes v_k^*].$$

Then F is a closed embedding, so to prove the Lemma, it suffices to identify the image with $\overline{\psi(T)}$. This is routine since by the above calculation, $\psi(T)$ is in the irreducible l-dimensional variety given by the image of F.

Q.E.D.

Note that $U^- \times U$ stabilizes $X_0 = X \cap \mathbb{P}_0$, and define $\chi : U^- \times U \times Z \to X_0$ by the formula $\chi(u, v, z) = (u, v) \cdot z = uzv^{-1}$.

Theorem 2.8. *χ is an isomorphism. In particular, X_0 is smooth, and moreover is isomorphic to $\mathbb{C}^{\dim(G)}$.*

Note that the second claim follows immediately from the first claim and the well-known fact that a unipotent algebraic group is isomorphic to its Lie algebra. We will prove several lemmas in order to prove the first claim in the Theorem.

The proof we give will essentially follow [4], and is due to DeConcini and Springer [7].

Lemma 2.9. *There exists a $U^- \times U$-equivariant morphism $\beta : X_0 \to U^- \times U$ such that $\beta \circ \chi(u, v, z) = (u, v)$ for all $z \in \psi(T)$.*

We assume this Lemma for now, and show how to use it to prove the Theorem. Recall the following easy fact.

Lemma 2.10. *Let A be an algebraic group and let Z be a A-variety, and regard A as a A-variety using left multiplication. Suppose there exists a A-equivariant morphism $f : Z \to A$. Let $F = f^{-1}(e)$. Then $Z \cong A \times F$.*

Proof. Define $\chi : A \times F$ by $\chi(a, z) = a \cdot z$, define $\eta : Z \to F$ by $\eta(z) = f(z)^{-1} \cdot z$, and define $\tau : Z \to A \times F$ by $\tau(z) = (f(z), \eta(z))$. Check that $\chi \circ \tau$ and $\tau \circ \chi$ are the identity.

Q.E.D.

Note also that there is an isomorphism $\sigma : G \times G \to G \times G$ given by $\sigma(x, y) = (y, x)$. Consider the isomorphism $\gamma : V \otimes V^* \to V^* \otimes V$ such that $\gamma(v \otimes f) = f \otimes v$. Then $\gamma \circ a = \sigma(a)\gamma$, for $a \in G \times G$.

Proof of Theorem 2.8. The last Lemma applied to Lemma 2.9 implies that

$$X_0 \cong (U^- \times U) \times \beta^{-1}(e,e). \tag{2.1}$$

But $\beta^{-1}(e,e)$ is closed in X_0 and by Lemma 2.9, $\psi(T) \subset \beta^{-1}(e,e)$. Thus, $Z = \overline{\psi(T)} \subset \beta^{-1}(e,e)$.

But X_0 is irreducible since X is irreducible, and $\dim(X_0) = \dim(U^- \times U) + l$, so by (2.1), $\beta^{-1}(e,e)$ is irreducible and l-dimensional. Thus, $\beta^{-1}(e,e) = Z$, and the proof of the Theorem follows.

Q.E.D.

Proof of 2.9. Define a morphism $F : \mathbb{P}_0 \to \mathbb{P}_0(V)$ by $F([A]) = [A \cdot v_0]$. The intrinsic definitions of $\mathbb{P}_0$ and $\mathbb{P}_0(V)$ imply that F is well-defined.

By restriction, we have $\phi : X_0 \to \mathbb{P}_0(V)$. By Lemma 2.6, $\psi(U^-TU) \subset X_0$. Let $u \in U^-, t \in T$, and $v \in U$. Then $\phi(\psi(utv)) = utv \cdot [v_0] = u \cdot [v_0]$ because TU fixes $[v_0]$. Thus, $\phi(\psi(U^-TU)) = U^- \cdot [v_0]$, which is closed in $\mathbb{P}_0(V)$ by Lemma 2.3. Since $\psi(U^-TU)$ is dense in X_0, it follows that $\phi(X_0) = U^- \cdot [v_0]$. It is easy to check that ϕ is U^--equivariant.

Define $\beta_1 : X_0 \to U^-$ by setting $\beta_1(x) = u$ if $u \cdot [v_0] = \phi(x)$. In other words, $\beta_1 = \eta \circ \phi$, where η is the inverse of the action isomorphism $U^- \to U^- \cdot [v_0]$ from Lemma 2.3. It follows from definitions that β_1 is U^--equivariant.

We now define a morphism $\beta_2 : X_0 \to U$. To do this, consider the identifications

$$\mathrm{End}(V) \cong V \otimes V^* \cong V^* \otimes V \cong \mathrm{End}(V^*)$$

where the second morphism is given by γ. By replacing B with B^-, we may regard v_0^* as a highest weight vector for V^*. Now the above result applied to V^* and B^- in place of V and B gives a morphism $J : X_0 \to \mathbb{P}_0(V^*)$ given by $J([A]) = A \cdot [v_0^*]$. Then reasoning as above, J restricts to a morphism $\psi : X_0 \to U \cdot [v_0^*]$ and then we set $\beta_2(x) = v \in U$ where $v \cdot [v_0^*] = \psi(x)$. As before, ψ is U-equivariant.

Now $\beta : X_0 \to U^- \times U$ given by $\beta(x) = (\beta_1(x), \beta_2(x))$ is the required morphism.

Q.E.D.

2.3. **Global Geometry.** We use the smoothness of the open set X_0 to show X is smooth.

Lemma 2.11. *Let K be a semisimple algebraic group and let V be an irreducible representation of K. Then the induced action of K on $\mathbb{P}(V)$ has a unique closed orbit through the line generated by a highest weight vector.*

Proof. We use the following standard facts:

(1) If H is a subgroup of K, then K/H is projective if and only if H is parabolic.

(2) A nonzero vector v in V is a highest weight vector if and only if its stabilizer $K_{[v]}$ in K is parabolic. If v_1, v_2 are highest weight vectors in V, then there is $g \in K$ such that $g \cdot v_1 = v_2$.

By (1) and (2), it is clear that if v is a highest weight vector, $K \cdot [v] \cong K/K_{[v]}$ is projective, and thus closed in $\mathbb{P}(V)$. Let $[u] \in \mathbb{P}(V)$ be such that $K \cdot [u] \cong K/K_{[u]}$ is closed and therefore projective. Then $K_{[u]}$ is parabolic by (1), so u is a highest weight vector. Thus, by (2), $K \cdot [u] = K \cdot [v]$.

Q.E.D.

Lemma 2.12. *Let A be an algebraic group and let W be a A-variety with a unique closed orbit Y. Let $V \subset W$ be open and suppose $V \cap Y$ is nonempty. Then $W = \cup_{g \in A}\, g \cdot V$.*

Proof. The union $\cup_{g \in A}\, g \cdot V$ is clearly open, so $W - \cup_{g \in A} g \cdot V$ is closed, and is A-stable. We assume it is nonempty and argue by contradiction. Recall that if an algebraic group acts on a variety, then it has a closed orbit. Thus, there exists a closed orbit Z for A on $W - \cup_{g \in A}\, g \cdot V$. Since Z is a closed orbit for A in W, then $Z = Y$. But $Y \not\subset W - \cup_{g \in A}\, g \cdot V$ by assumption. It follows that $W - \cup_{g \in A} g \cdot V$ is empty.

Q.E.D.

Lemma 2.13. *Let $W \subset X = \overline{\psi(G)}$ be a $G \times G$-stable closed subvariety of X. Then $W = \cup_{a \in G \times G}\, a \cdot (W \cap X_0)$.*

Proof. Note that $\mathrm{End}(V) \cong V \otimes V^*$ is an irreducible representation of the semisimple group $G \times G$ and $v_0 \otimes v_0^*$ is a highest weight vector stabilized by $B \times B^-$. Thus, by Lemma 2.11, $\mathbb{P}(\mathrm{End}(V))$ has a unique closed $G \times G$-orbit through $p_\lambda := [v_0 \otimes v_0^*]$. Since $W \subset \mathbb{P}(\mathrm{End}(V))$ is $G \times G$-stable, it has a closed orbit, which must be projective since W is projective. This closed orbit must be $Y = (G \times G) \cdot p_\lambda$. Since $p_\lambda \subset X_0 \cap Y$, $X_0 \cap W$ is nonempty. Thus, by Lemma 2.12, the conclusion follows.

Q.E.D.

Proposition 2.14. *The following hold:*

(1) *$X = \cup_{a \in G \times G}\, a \cdot X_0$ is smooth.*
(2) *Let $Q \subset X$ be a $G \times G$-orbit. Then*

$$\overline{Q} = \cup_{a \in G \times G}\, a \cdot (\overline{Q} \cap X_0).$$

(3) *If Q and Q' are two $G \times G$-orbits in X, and $\overline{Q} \cap X_0 = \overline{Q'} \cap X_0$, then $Q = Q'$.*

Proof. For (2), apply Lemma 2.13 to the case $W = \overline{Q}$. This gives (2), and in the case $Q = \psi(G)$, we obtain $\overline{Q} = X$, which gives

$$X = \cup_{a \in G \times G}\, a \cdot X_0.$$

Now (1) follows by Theorem 2.8 since X is a union of smooth open sets. To prove (3), note that by (2), $\overline{Q} = \overline{Q'}$. Since Q and Q' are both open in their closures, they coincide.

Q.E.D.

2.4. **Description of the $G \times G$-orbits.** We classify the $G \times G$-orbits in X and show they have smooth closure.

For $I \subset \{1, \ldots, l\}$, let

$$Z_I := \{(z_1, \ldots, z_l) \in Z : z_i = 0 \ \forall i \in I\}$$

and let

$$Z_I^0 := \{(z_1, \ldots, z_l) \in Z_I : z_j \neq 0 \ \forall j \notin I\}.$$

Then Z_I is the closure of Z_I^0, $Z_I \cong \mathbb{C}^{l-|I|}$ and $Z_I^0 \cong (\mathbb{C}^*)^{l-|I|}$. Moreover, by the proof of Lemma 2.7, $T \cong (\mathbb{C}^*)^l$ acts on $Z \cong \mathbb{C}^l$ by $(a_1, \ldots, a_l) \cdot (z_1, \ldots, z_l) = (a_1 z_1, \ldots, a_l z_l)$ in appropriate coordinates, so Z is $(T \times \{e\})$-stable, and the Z_I^0 are exactly the $T \times \{e\}$-orbits in Z.

The next lemma follows from the above discussion.

Lemma 2.15. *Set $z_I = (z_1, \ldots, z_l)$, where $z_i = 1$ if $i \notin I$, and $z_i = 0$ if $i \in I$. Then $Z_I^0 = (T \times \{e\}) \cdot z_I$.*

For $i = 1, \ldots, l$, $Z_i := Z_{\{i\}}$ is a hypersurface, and $Z_I = \cap_{i \in I} Z_i$.
Let

$$\Sigma_I = \chi(U^- \times U \times Z_I)$$

and let

$$\Sigma_I^0 = \chi(U^- \times U \times Z_I^0)$$

The following result follows easily from Theorem 2.8.

Proposition 2.16. *The Σ_I^0 are precisely the $U^-T \times U$ orbits in X_0, and the closure Σ_I of Σ_I^0 in X_0 is isomorphic to $\mathbb{C}^{\dim(G)-|I|}$. In particular, $U^-T \times U$ has precisely 2^l orbits on X_0, and all these orbits have smooth closure.*

Remark 2.17. *Let Q be a $G \times G$-orbit in X. Then $\overline{Q} \cap X_0$ is closed, irreducible, and $U^-T \times U$-stable, so it must be the closure of one of the 2^l $U^-T \times U$-orbits in X_0.*

By this last remark and Proposition 2.14 (3), it follows that $G \times G$ has at most 2^l orbits on X. We show there are exactly 2^l $G \times G$ orbits.

For the following result, see [13], chapter II.

Lemma 2.18. *Let W be an irreducible projective variety with open set U. Then if U is isomorphic to an affine variety, all irreducible components of $W - U$ have codimension 1.*

Lemma 2.19. *$X - \psi(G) = \cup_{i=1,\ldots,l} S_i$, where*

1. *S_i is a $G \times G$-stable divisor*
2. *$S_i = \overline{\Sigma_i}$, and*
3. *$S_i \cap X_0 = \Sigma_i$.*

Proof. Let $X - \psi(G) = \cup_\alpha S_\alpha$ be the decomposition of $X - \psi(G)$ into irreducible components. Since $\psi(G) \cong G$ is affine, each S_α has codimension one by Lemma 2.18. Since $X - \psi(G)$ is $G \times G$-stable, and $G \times G$ is connected, each S_α is $G \times G$ stable. Thus, $S_\alpha \cap X_0$ is $U^-T \times U$-stable and is a closed irreducible hypersurface in X_0 by Lemma 2.13. Since X_0 has finitely many $U^-T \times U$-orbits, it must be one of the Σ_i. Conversely, the closure of Σ_i in X is a closed, irreducible hypersurface. Since $X_0 \cap \psi(G) = \Sigma_\emptyset$ by Lemma 2.6, Σ_i is contained in the closed set $X - \psi(G)$, so $\overline{\Sigma_i}$ is contained in $X - \psi(G)$. Thus, $\overline{\Sigma_i}$ must be an irreducible component of $X - \psi(G)$, so $\overline{\Sigma_i} = S_\alpha$ for some α, and $S_\alpha \cap X_0 = \Sigma_i$. The result follows.

Q.E.D.

Lemma 2.20. *The following hold:*

(1) *Let $S_I = \cap_{i \in I} S_i$. Then $S_J \subset S_I$ if $I \subset J$ and $S_I \cap X_0 = \Sigma_I$.*
(2) *Let $S_I^0 = S_I - \cup_{I \subsetneq J} S_J$. Then $S_I^0 = (G \times G) \cdot z_I$ is a single $G \times G$-orbit and $S_I^0 = S_J^0$ implies $I = J$.*
(3) *$S_I = \cup_{a \in G \times G}\ a \cdot \Sigma_I$, and in particular S_I is smooth.*

Proof. The first claim of (1) is obvious from the definition and the second claim follows from Lemma 2.19(3). For (2), it is clear that $S_I^0 \cap X_0 = \Sigma_I^0$. Since the S_i are $G \times G$-stable, S_I is $G \times G$-stable, so S_I^0 is $G \times G$-stable. Let $x, y \in S_I^0$. By Proposition 2.14 (1), there are a and b in $G \times G$ such that $a \cdot x$ and $b \cdot y$ are in X_0. Since $a \cdot x$ and $b \cdot y$ are in S_I^0 by $G \times G$-stability, they are in Σ_I^0. Thus, by Proposition 2.16, there is $c \in U^-T \times U$ such that $a \cdot x = cb \cdot y$. Part (2) follows since clearly $z_I \in S_I^0$. Part (3) follows by the above assertions and Proposition 2.14(2).

Q.E.D.

Remark 2.21. *Recall the following definition. Let X be a smooth variety with hypersurface Z. We say Z is a divisor with normal crossings at $x \in Z$ if there is an open neighborhood U of x such that $Z \cap U = D_1 \cup \cdots \cup D_k$ is a union of hypersurfaces and $D_{i_1} \cap \cdots \cap D_{i_j}$ is smooth of codimension j for each distinct j-tuple $\{i_1, \ldots, i_j\}$ in $\{1, \ldots, k\}$. A standard example is to take $X = \mathbb{C}^n$ and Z to be the variety defined by the vanishing of $z_1 \ldots z_k$. Then Z is the union of the hyperplanes given by vanishing of z_i, and in the complex analytic setting, every divisor with normal crossings is locally of this nature.*

Theorem 2.22. *$G \times G$ has 2^l orbits in X, given by $S_I^0 = (G \times G) \cdot z_I$ where I is a subset of $\{1, \ldots, l\}$. In particular, all orbits have smooth closure, and the pair $(X, X - \psi(G))$ is a divisor with normal crossings.*

Proof. Let $Q \subset X$ be a $G \times G$-orbit. Then $\overline{Q} \cap X_0 = \Sigma_I$ for some $I \subset \{1, \ldots, l\}$ by Remark 2.17 and Proposition 2.16. It follows easily that $\Sigma_I^0 \subset Q$ so $S_I^0 = Q$, using Lemma 2.20 (2). Again by Lemma 2.20 (2), there are exactly 2^l orbits. Since Σ_I is the closure of Σ_I^0 in X_0, it follows that S_I is the closure of S_I^0 in X using Lemma 2.12. It is clear from Propositon 2.16 that $(X_0, X_0 - \psi(U^-TU))$ is a divisor with normal crossings. The last assertion follows from Proposition 2.14 (1) and group action invariance of the divisor with normal crossings property.

Q.E.D.

2.5. **Geometry of orbits and their closures.** We now want to understand the geometry of S_I^0 and S_I. We show that the orbit S_I^0 fibers over a product of generalized flag varieties, with fiber a semisimple group. In this picture, the closure S_I fibers over the same product of generalized flag varieties, and the fiber is the wonderful compactification of the semisimple group.

For a subset $I \subset \{1, \ldots, l\}$, let $\Delta_I = \{\alpha_i : i \notin I\}$. Let Φ_I be the roots that are in the linear span of the simple roots in Δ_I, and let

$$\mathfrak{l}_I = \mathfrak{t} + \sum_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$$

Let $\mathfrak{p}_I = \mathfrak{l}_I + \mathfrak{u}$ and let $\mathfrak{p}_I^- = \mathfrak{l}_I + \mathfrak{u}^-$. Let $\mathfrak{u}_I$ and $\mathfrak{u}_I^-$ be the nilradicals of $\mathfrak{p}_I$ and $\mathfrak{p}_I^-$, respectively. Note that $\mathfrak{p}_\emptyset = \mathfrak{g}$ and $\mathfrak{p}_{\{1,\ldots,l\}} = \mathfrak{b}$. Let P_I, P_I^-, U_I, U_I^-, and L_I be the corresponding connected subgroups of G. Let $Z(L_I)$ be the center of L_I and let $G_I = L_I/Z(L_I)$ be the adjoint group of L_I and denote its Lie algebra by $\mathfrak{g}_I$.

For a Lie algebra $\mathfrak{a}$, let $U(\mathfrak{a})$ be its enveloping algebra. Let $V_I = U(\mathfrak{l}_I) \cdot v_0$. Then V_I is a $\mathfrak{l}_I$-stable submodule of V and V_I may be identified with the irreducible representation of $\mathfrak{g}_I$ of highest weight λ.

Lemma 2.23. *The following hold:*

(1) *V_I is $\mathfrak{p}_I$-stable, and $\mathfrak{u}_I$ annihilates V_I.*
(2) *Let $Q = \{g \in G : g \cdot V_I = V_I\}$. Then $Q = P_I$.*

Proof. For (1), by the Poincare-Birkhoff-Witt Theorem, $U(\mathfrak{p}_I) = U(\mathfrak{l}_I)U(\mathfrak{u}_I)$, and $\mathfrak{p}_I$-stability of V_I follows by the theorem of the highest weight. Since $\mathfrak{u}_I$ is an ideal of $\mathfrak{p}_I$, the space of $\mathfrak{u}_I$-invariants $V^{\mathfrak{u}_I}$ is $\mathfrak{l}_I$-stable and meets V_I. Since it is nonzero and V_I is an irreducible $\mathfrak{l}_I$-module, $V_I = V^{\mathfrak{u}_I}$.

For (2), note that it follows from (1) that $P_I \subset Q$. Hence Q is parabolic so it is connected. To show $P_I = Q$, it suffices to show $\mathfrak{p}_I = \mathfrak{q}$, where $\mathfrak{q}$ is the Lie algebra of Q. There is $t \in Z(L_I)$ such that $\alpha_i(t) \neq 1$ for every simple root $\alpha_i \notin \Delta_i$. Then if $\tilde{t} \in \tilde{T}$ is a preimage of t, $Ad_{\tilde{t}}$ acts by $\lambda(\tilde{t})$ on V_I, and if $x \in \mathfrak{g}_{-\alpha_i}$ for α_i simple, $Ad_{\tilde{t}}$ has weight $\frac{\lambda(\tilde{t})}{\alpha_i(t)}$ on $x \cdot V_I$. It follows that if $\alpha_i \notin \Delta_i$, $\mathfrak{g}_{-\alpha_i} \cdot V_I \cap V_I = 0$. Since λ is regular, $\mathfrak{g}_{-\alpha_i} \cdot v_0 \neq 0$ for each simple root α_i, so $\mathfrak{g}_{-\alpha_i} \cdot v_0 \notin V_I$ if $\alpha_i \notin \Delta_I$. It follows that $\mathfrak{g}_{-\alpha_i} \not\subset \mathfrak{q}$ for all $\alpha_i \notin \Delta_I$, so $\mathfrak{q} = \mathfrak{p}_I$.

Q.E.D.

Lemma 2.24. *The following hold:*

(1) *Let*
$$J = \{k \in \{0, \ldots, n\} : \lambda_k = \lambda - \sum_{i \in \Delta_I} n_i \alpha_i, n_i \geq 0\}$$
Then the $\{v_j : j \in J\}$ *form a basis of* V_I.

(2) *Recall* $z_I = \{(\epsilon_1, \ldots, \epsilon_l) \in Z_I\}$, *where* $\epsilon_i = 1$ *if* $i \notin I$ *and* $\epsilon_i = 0$ *if* $i \in I$. *Then* $z_I = [\mathrm{pr}_{V_I}]$, *where* pr_{V_I} *is the projection on* V_I *such that* $\mathrm{pr}_{V_I}(v_k) = 0$ *if* $k \notin J$.

Proof. (1) is routine and is left to the reader. For (2), recall that for $\{z_1, \ldots, z_l\} \in Z$, the corresponding class in $\mathbb{P}(\mathrm{End}(V))$ is
$$[v_0 \otimes v_0^* + \sum_{i=1}^{l} z_i v_i \otimes v_i^* + \sum_{k>l} \prod_{i=1}^{l} z_i^{n_{ik}} v_k \otimes v_k^*]$$
where $\lambda_k = \lambda - \sum_{i=1}^{l} n_{ik} \alpha_i$. It follows that
$$z_I = [\sum_{k \in J} v_k \otimes v_k^*].$$
But $\sum_{k \in J} v_k \otimes v_k^*$ is easily identified with pr_{V_I}.

Q.E.D.

We now compute the stabilizer of $(G \times G)$ at the point $z_I \in S_I^0 = (G \times G) \cdot z_I$.

Proposition 2.25. *The stabilizer* $(G \times G)_{z_I}$ *of* z_I *in* $G \times G$ *is*
$$\{(xu, yv) : u \in U_I, v \in U_I^-, x \in L_I, y \in L_I, \text{ and } xy^{-1} \in Z(L_I)\}$$
In particular, S_I^0 *fibers over* $G/P_I \times G/P_I^-$ *with fiber* G_I.

Proof. Suppose $(r, s) \cdot z_I = z_I$ for $(r, s) \in G \times G$. Then $[r \mathrm{pr}_{V_I} s^{-1}] = [\mathrm{pr}_{V_I}]$ so r preserves the image V_I of pr_{V_I}. Thus, by Lemma 2.23, $r \in P_I$. Further, if $r \in U_I$, then $r \cdot [\mathrm{pr}_{V_I}] = \mathrm{pr}_{V_I}$ since U_I acts trivially on V_I.

Recall the identification $\gamma : \mathrm{End}(V) \to \mathrm{End}(V^*)$ from the proof of 2.9. Let $V_I^* = U(\mathfrak{p}^-) v_0^* \subset V^*$. V_I^* is an irreducible representation of L_I with lowest weight $-\lambda$. One can easily check that $\gamma(\mathrm{pr}_{V_I}) = \mathrm{pr}_{V_I^*}$, and $\gamma((x, y) \cdot A) = (y, x) \cdot \gamma(A)$, for $(x, y) \in G \times G$ and $A \in \mathrm{End}(V)$. Hence
$$(s, r) \cdot [\mathrm{pr}_{V_I^*}] = \gamma((r, s) \cdot [\mathrm{pr}_{V_I}]) = \gamma([\mathrm{pr}_{V_I}]) = [\mathrm{pr}_{V_I^*}].$$
As above, it follows that s preserves V_I^*. Thus, by Lemma 2.23 applied to the opposite parabolic, $s \in P_I^-$. Further, if $s \in U_I^-$, then $[\mathrm{pr}_{V_I} \circ s] = [\mathrm{pr}_{V_I}]$.

Now let $r = xu$ and let $s = yv$ with $x, y \in L_I$, $u \in U_I$ and $v \in U_I^-$. Then
$$r \cdot [\mathrm{pr}_{V_I}] \cdot s^{-1} = x \cdot [\mathrm{pr}_{V_I}] \cdot y^{-1}.$$

In particular, xy^{-1} acts trivially on $\mathbb{P}(V_I)$. But

$$Z(L_I) = \{g \in L_I : g \cdot [v] = [v], \forall [v] \in \mathbb{P}(V_I)\},$$

so $xy^{-1} \in Z(L_I)$. Now we consider the projection $(G\times G)/(G\times G)_{z_I} \to (G\times G)/(P_I \times P_I^-)$ with fiber

$$(P_I \times P_I^-)/(G \times G)_{z_I} \cong (L_I \times L_I)/\{(x,y) : x \in L_I, y \in L_I, \text{ and } xy^{-1} \in Z(L_I)\}$$

The morphism

$$(L_I \times L_I)/\{(x,y) : x \in L_I, y \in L_I, \text{ and } xy^{-1} \in Z(L_I)\} \to L_I/Z(L_I)$$

given by $(a,b) \mapsto ab^{-1}$ is easily seen to be an isomorphism.

Q.E.D.

To understand the closure S_I of S_I^0, we embed the compactification of a smaller group into X.

First, note that we may embed $\mathrm{End}(V_I)$ into $\mathrm{End}(V)$ by using the map

$$\sum_{i,j\in J} a_{ij} v_i \otimes v_j^* \to \sum_{i,j\in\{0,\dots,n\}} b_{ij} v_i \otimes v_j^*$$

where $b_{ij} = a_{ij}$ if $i,j \in J$, and $b_{ij} = 0$ otherwise.

This map induces an embedding $\mathbb{P}(\mathrm{End}(V_I)) \to \mathbb{P}(\mathrm{End}(V))$, and we will regard $\mathbb{P}(\mathrm{End}(V_I))$ as a closed subvariety of $\mathbb{P}(\mathrm{End}(V))$.

It follows from definitions that $z_K \in \mathbb{P}(\mathrm{End}(V_I)) \iff I \subset K$.

Define a morphism $L_I \to \mathbb{P}(\mathrm{End}(V))$ by $\psi_I(g) = [g\mathrm{pr}_{V_I}]$. Then ψ_I descends to a morphism $\psi_I : L_I/Z(L_I) = G_I \to \mathbb{P}(\mathrm{End}(V))$. Note that the image of ψ_I is in $\mathbb{P}(\mathrm{End}(V_I))$.

Recall that the center of G_I is trivial, and note that V_I is an irreducible representation of a cover of G_I with regular highest weight. We set $X_I = \overline{\psi_I(G_I)}$, and we may apply our results about orbit structure for the compactification X of G to the compactification X_I of G_I. In particular, X_I is a $G_I \times G_I$-variety and we may regard X_I as a $P_I \times P_I^-$-variety via the projection $P_I \times P_I^- \to (P_I \times P_I^-)/(Z(L_I)U_I \times Z(L_I)U_I^-) = G_I \times G_I$.

Theorem 2.26. *Consider the morphism*

$$\chi : (G \times G) \times_{P_I \times P_I^-} X_I \to S_I, \ \chi(g_1, g_2, x) = (g_1, g_2) \cdot x.$$

Then, χ is an isomorphism of varieties. In particular, S_I fibers over $G/P_I \times G/P_I^-$ with fiber X_I.

Proof. It suffices to prove χ is a bijection, since a bijection to a smooth variety in characteristic zero is an isomorphism. For $I \subset K$, let $y_K = (e, e, z_K) \in (G\times G) \times_{P_I \times P_I^-} X_I$ and note that $\chi(y_K) = z_K$. Since χ is $G \times G$-equivariant, it follows that χ is surjective by Theorem 2.22.

To show χ is injective, we use the formal fact:

(*) Let A be a group with subgroup B. Let Y be a B-set. Then B-orbits in Y correspond to A-orbits in $A \times_B Y$ via $B \cdot y \mapsto A \cdot (e, y)$. Moreover, the stabilizer B_y coincides with the stabilizer $A_{(e,y)}$.

It follows that the stabilizer $(G \times G)_{y_K} = (P_I \times P_I^-)_{z_K}$. But $U_I Z(L_I) \times U_I^- Z(L_I)$ acts trivially on X_I since it fixes each z_K with $I \subset K$, so $(P_I \times P_I^-)_{z_K}$ is the preimage of $(G_I \times G_I)_{z_K}$ in $P_I \times P_I^-$. By Proposition 2.25 applied to G_I and X_I, it follows that

$$(P_I \times P_I^-)_{z_K} = \{(xu, yv) : u \in U_K, v \in U_K^-, x \in L_K, y \in L_K, \text{ and } xy^{-1} \in Z(L_K)\},$$

which coincides with the stabilizer $(G \times G)_{z_K}$. Thus, χ is injective when restricted to a $G \times G$ orbit.

To complete the proof that χ is injective, it suffices to show that every $G \times G$-orbit contains some y_K, $I \subset K$. This is an easy consequence of (*) and the classification of G_I orbits in its wonderful compactification X_I, which follows from Theorem 2.22. We leave the routine details to the reader.

Q.E.D.

3. Comparison of different compactifications

3.1. **Independence of highest weight.** For completeness, we show that X is independent of the choice of the regular, dominant highest weight. This result is not especially surprising since λ does not appear in the statements describing the $G \times G$-orbit structure. The proof mostly follows that of DeConcini and Springer ([7], Proposition 3.10).

We may carry out the wonderful compactification construction using regular, dominant weights λ_1 and λ_2 to get varieties $X^1 \subset \mathbb{P}(\mathrm{End}V(\lambda_1))$ and $X^2 \subset \mathbb{P}(\mathrm{End}V(\lambda_2))$ respectively. We denote the class of the identity by $[id_1] \in X^1$ and $[id_2] \in X^2$. We define $X^\Delta = \overline{(G \times G) \cdot ([id_1], [id_2])} \subset X^1 \times X^2$, and prove that the natural projections $p_1 : X^\Delta \to X^1$ and $p_2 : X^\Delta \to X^2$ are isomorphisms.

Proposition 3.1. *For $i = 1, 2$, $p_i : X^\Delta \to X^i$ is a $G \times G$-equivariant isomorphism of varieties. In particular, $p_2 \circ p_1^{-1} : X^1 \to X^2$ is a $G \times G$-equivariant isomorphism sending $[id_1]$ to $[id_2]$.*

In the proof, we consider constructions in X^i analogous to constructions used in Section 2, such as the open affine piece, and the closure of T in the open affine piece. We denote the various subsets defined in Section 2 for X^i with a superscript i, e.g., $Z^i \subset X_0^i = \mathbb{P}_0(\mathrm{End}V(\lambda_i)) \cap X^i$.

First, let $Z^\Delta := \overline{(T \times \{e\}) \cdot ([id_1], [id_2])}$ where closure is taken in the open set $X_0^1 \times X_0^2 \subset X^1 \times X^2$.

Lemma 3.2. *$Z^\Delta \cong \mathbb{C}^l$ and $p_i : Z^\Delta \to Z^i$ is a $(T \times \{e\})$-equivariant isomorphism.*

Proof. This is a straightforward calculation using coordinates as in Lemma 2.7. Choose a basis of weight vectors $\{v_i\}_{i=0..n}$ for $V(\lambda_1)$ and a basis of weight vectors $\{w_i\}_{i=0..m}$ for

$V(\lambda_2)$ satisfying properties (1)-(3) preceding Remark 2.1. Then, for $t \in T$,

$$\begin{aligned}(t \times \{e\}) \cdot ([id_1], [id_2]) &= (t \times \{e\}) \cdot ([\textstyle\sum_{i=0}^n v_i^* \otimes v_i], [\sum_{i=0}^m w_i^* \otimes w_i]) \\ &= ([v_0^* \otimes v_0 + \textstyle\sum_{i=1}^l \frac{1}{\alpha_i(t)} v_i^* \otimes v_i + \cdots], [w_0^* \otimes w_0 + \sum_{i=1}^l \frac{1}{\alpha_i(t)} w_i^* \otimes w_i + \cdots])\end{aligned}$$

where the terms indicated by $\cdots$ in the two factors have coefficients which are polynomial in the $\frac{1}{\alpha_i(t)}$. Then the map sending $(z_1, \ldots, z_l) \in \mathbb{C}^l$ to the above expression with $\frac{1}{\alpha_i(t)}$ replaced by z_i identifies Z^Δ with $\mathbb{C}^l$ and the claim follows.

Q.E.D.

Recall the embedding from Theorem 2.8,

$$\chi^i : U^- \times U \times Z^i \to X^i$$

This is an isomorphism to X_0^i. Let $X_0^\Delta := p_i^{-1}(X_0^i) \subset X^\Delta$, and consider the subset $V = \bigcup_{a \in G \times G} a \cdot X_0^\Delta$ of X^Δ. Define $\chi^\Delta : U^- \times U \times Z^\Delta \to X^\Delta$ in the same manner as χ from Theorem 2.8.

We have a commutative diagram:

$$\begin{tikzcd} U^- \times U \times Z^\Delta \arrow[r, "\chi^\Delta"] \arrow[d, "id \times id \times p_i", "\cong"'] & X^\Delta \arrow[d, "p_i"] \\ U^- \times U \times Z^i \arrow[r, "\chi^i"] & X^i. \end{tikzcd} \tag{3.1}$$

We will use the following two lemmas, which we prove below.

Lemma 3.3. *χ^Δ is an embedding onto the open set X_0^Δ.*

Lemma 3.4. *$p_i|_V$ is injective.*

We assume these for now and prove that p_i is an isomorphism.

Proof of Proposition 3.1. Consider the commutative diagram:

$$\begin{tikzcd} U^- \times U \times Z^\Delta \arrow[r, "\chi^\Delta"] \arrow[d, "id \times id \times p_i", "\cong"'] & X_0^\Delta \arrow[d, "p_i"] \\ U^- \times U \times Z^i \arrow[r, "\chi^i"] & X_0^i. \end{tikzcd} \tag{3.2}$$

Since $\chi^i : U^- \times U \times Z^i \to X_0^i$ is surjective, $p_i : X_0^\Delta \to X_0^i$ is surjective using the above commutative diagram. Then $p_i : V \to X^i$ is surjective by Proposition 2.14 (1). Further, $p_i : V \to X^i$ is injective by Lemma 3.4, and hence an isomorphism since X^i is smooth.

Thus V is complete, so $V = X^\Delta$, since X^Δ is irreducible and $\dim(V) = \dim(X^\Delta)$.

Q.E.D.

We now prove the two lemmas.

Proof of Lemma 3.3. First, χ^Δ is an embedding since $\chi^i \circ (id \times id \times p_i|_{Z^\Delta})$ is an embedding and the diagram (3.1) commutes.

Let Y denote the image of χ^Δ. Commutativity of (3.1) and Theorem 2.8 imply that $Y \subset X_0^\Delta$. Consider the map

$$s = \chi^\Delta \circ (id \times id \times p_i)^{-1} \circ (\chi^i)^{-1} : X_0^i \to X_0^\Delta.$$

Note that s is a section for the map p_i over X_0^i, i.e., $p_i \circ s = id$. Consider the composition $f = s \circ p_i|_{X_0^\Delta}$. It is routine to check that $f|_Y = id|_Y$. Y and X_0^Δ have the same dimension and X_0^Δ is irreducible since it is an open subset of the irreducible variety X^Δ. Hence, there is an open, dense subset of X_0^Δ contained in Y. Then f is the identity on an open, dense subset of X_0^Δ, so f is the identity on X_0^Δ. Hence, $Y = X_0^\Delta$.

Q.E.D.

Proof of Lemma 3.4. For $I \subset \{1, \ldots, l\}$, let $z_I^\Delta = (z_I^1, z_I^2)$, so $p_i(z_I^\Delta) = z_I^i$. Then the stabilizer $(G \times G)_{z_I^\Delta} = (G \times G)_{z_I^1} \cap (G \times G)_{z_I^2}$, which coincides with $(G \times G)_{z_I^i}$ for $i = 1, 2$, since $(G \times G)_{z_I^1} = (G \times G)_{z_I^2}$ by Proposition 2.25. Thus, $p_i : V \to X^i$ is injective when restricted to the orbits through some z_I^Δ. By Lemma 3.3, each $U^-T \times U$-orbit on X_0^Δ meets Z^Δ, so it meets some z_I^Δ. The Lemma follows.

Q.E.D.

We now prove a result related to Proposition 3.1 which will be used in the sequel.

Let E be a representation of $\tilde{G} \times \tilde{G}$. Then the $\tilde{G} \times \tilde{G}$ action on $\mathbb{P}(E)$ descends to an action of $G \times G$ on $\mathbb{P}(E)$. Suppose there exists a point $[x] \in \mathbb{P}(E)$ such that $(G \times G)_{[x]} = G_\Delta$. We may embed G into $\mathbb{P}(E)$ by the mapping $\psi : G \to \mathbb{P}(E)$ given by $\psi(g) = (g, e) \cdot [x]$. Let $X(E, [x])$ be the closure of $\psi(G)$ in $\mathbb{P}(E)$. Let $X_\lambda = X(\mathrm{End}(V), [id_V])$ when V is irreducible of highest weight λ. If λ is regular, then X_λ is of course smooth and projective with known $G \times G$-orbit structure by Theorem 2.22.

Let $W_1, \ldots, W_k$ be a collection of irreducible representations of G of highest weights $\mu_1, \ldots, \mu_k$. Let $W = W_1 \oplus \cdots \oplus W_k$. Let F be a representation of $\tilde{G} \times \tilde{G}$. Let V have highest weight λ as before. When useful, we will denote the irreducible representation of highest weight λ by $V(\lambda)$.

Proposition 3.5. *Suppose each μ_j is of the form $\mu_j = \lambda - \sum n_i \alpha_i$ with all n_i nonnegative integers. Then,*

$$X(\mathrm{End}(V) \oplus \mathrm{End}(W) \oplus F, [\mathrm{id}_V \oplus \mathrm{id}_W \oplus 0]) \cong X_\lambda.$$

Proof. It is immediate from definitions that

$$X(\mathrm{End}(V) \oplus \mathrm{End}(W) \oplus F, [\mathrm{id}_V \oplus \mathrm{id}_W \oplus 0]) \cong X(\mathrm{End}(V) \oplus \mathrm{End}(W), [\mathrm{id}_V \oplus \mathrm{id}_W]),$$

i.e., nothing is lost by setting $F = 0$. Indeed, the $G \times G$-orbit through $[\mathrm{id}_V \oplus \mathrm{id}_W \oplus 0]$ lies inside the closed subvariety $\mathbb{P}(\mathrm{End}(V) \oplus \mathrm{End}(W))$ of $\mathbb{P}(\mathrm{End}(V) \oplus \mathrm{End}(W) \oplus F)$, and this implies the claim.

Define $X' = X(\mathrm{End}(V) \oplus \mathrm{End}(W), [\mathrm{id}_V \oplus \mathrm{id}_W])$ for notational simplicity.

Consider the open subset $\tilde{\mathbb{P}}(\mathrm{End}(V) \oplus \mathrm{End}(W)) = \{[A \oplus B] : A \neq 0\}$ of $\mathbb{P}(\mathrm{End}(V) \oplus \mathrm{End}(W))$. The projection $\pi : \tilde{\mathbb{P}}(\mathrm{End}(V) \oplus \mathrm{End}(W)) \to \mathbb{P}(\mathrm{End}(V))$ given by $[A \oplus B] \mapsto [A]$ is a morphism of varieties.

We claim that $\tilde{\mathbb{P}}(\mathrm{End}(V) \oplus \mathrm{End}(W))$ is $(G \times G)$-stable. Indeed, if $A \neq 0$ and $(x, y) \in \tilde{G} \times \tilde{G}$, then xAy^{-1} is nonzero since x and y are invertible. It follows that if $[A \oplus B] \in \tilde{\mathbb{P}}(\mathrm{End}(V) \oplus \mathrm{End}(W))$, then $(x, y) \cdot [A \oplus B] \in \tilde{\mathbb{P}}(\mathrm{End}(V) \oplus \mathrm{End}(W))$, which establishes the claim.

Let $w_0(j)$ be a highest weight vector for W_j, and let $w_0(j)^*$ be a nonzero vector of weight $-\mu_j$ of W_j^*, normalized so $w_0(j)^*(w_0(j)) = 1$. Let $w_0 = \sum_{j=1}^k w_0(j)$ and let $w_0^* = \sum_{j=1}^k w_0(j)^*$. Then $w_0^*(w_0) = k$. Define an open subset $\mathbb{P}_0(\mathrm{End}(V) \oplus \mathrm{End}(W))$ as the set of $[A \oplus B]$ such that $v_0^*(A \cdot v_0) \neq 0$ and $w_0^*(B \cdot w_0) \neq 0$. It is easy to see that $\mathbb{P}_0(\mathrm{End}(V) \oplus \mathrm{End}(W))$ is $T \times T$-stable.

Let Z' be the closure of $\psi(T)$ in $\mathbb{P}_0(\mathrm{End}(V) \oplus \mathrm{End}(W))$.

Claim: $Z' \cong \mathbb{C}^l$.

The proof of this claim is essentially the same as the proof of Lemma 2.7. Indeed, we may compute $\psi(t) = (t, e) \cdot [\mathrm{id}_V \oplus \mathrm{id}_W]$ in the same manner as in Lemma 2.7. In 2.7, the coordinates z_i are essentially given by $\frac{1}{\alpha_i(t)}$ and the assumption that $\mu_j \leq \lambda$ implies that the additional summands that appear in $(t, e) \cdot [\mathrm{id}_V \oplus \mathrm{id}_W]$ have coefficients that are polynomial in the z_i. We leave details to the reader.

It is easy to see that $Z' \subset \tilde{\mathbb{P}}(\mathrm{End}(V) \oplus \mathrm{End}(W))$, and in fact $\pi : Z' \to Z$ is an isomorphism compatible with the identifications with $\mathbb{C}^l$. It follows from $(G \times G)$-stability of $\tilde{\mathbb{P}}(\mathrm{End}(V) \oplus \mathrm{End}(W))$ that $X_1' := (G \times G) \cdot Z'$ is in $\tilde{\mathbb{P}}(\mathrm{End}(V) \oplus \mathrm{End}(W))$.

By Theorem 2.22, $X = (G \times G) \cdot Z$. It follows that $\pi : X_1' \to X$ is surjective.

We claim that $\pi : X_1' \to X$ is injective, so that $X' \cong X$ since X is smooth. It follows that X_1' is projective since X is projective, so X_1' is closed in X', and has dimension equal to the dimension of G. But X' is irreducible of dimension equal to the dimension of G so $X' = X_1'$.

For $I \subset \{1, \dots, l\}$, define z_I' by the same formula as in Lemma 2.15. It is routine to show that $Z' = \cup_I \, (T, e) \cdot z_I'$ and $\pi(z_I') = z_I$. To show that π is injective, it suffices to check that the stabilizer $(G \times G)_{z_I'} = (G \times G)_{z_I}$. From $(G \times G)$-equivariance of π, it follows that $(G \times G)_{z_I'} \subset (G \times G)_{z_I}$. We computed $(G \times G)_{z_I}$ in Proposition 2.25. Using this computation, it is not difficult to check that $(G \times G)_{z_I} \subset (G \times G)_{z_I'}$. Indeed, we have $z_I' = [\mathrm{pr}_{V_I} + \sum_{j=1}^k \mathrm{pr}_{W_{j,I}}]$, where $W_{j,I}$ is $U(\mathfrak{l}_I) \cdot w_0(j)$. By Proposition 2.25, the stabilizer

$(G \times G)_{z_I}$ acts by scalars on $U(\mathfrak{l}_I) \cdot w_0(j)$. This implies injectivity of π, and completes the proof of the Proposition.

Q.E.D.

Remark 3.6. *In the statement of Proposition 3.5, we can replace* $\mathrm{End}(W)$ *with* $\mathrm{End}(W_1) \oplus \cdots \oplus \mathrm{End}(W_k)$*, and we can replace* id_W *with* $c_1\mathrm{id}_{W_1} \oplus \cdots \oplus c_k\mathrm{id}_{W_k}$*, where* $c_1, \dots, c_k$ *are scalars. Indeed, when all* $c_j = 1$*, this follows because* $\mathrm{End}(W_1) \oplus \cdots \oplus \mathrm{End}(W_k)$ *is a* $(G \times G)$*-submodule of* $\mathrm{End}(W)$*, so we may compute the closure in the smaller space. Moreover, in this embedding,* $\mathrm{id}_W = \mathrm{id}_{W_1} \oplus \cdots \oplus \mathrm{id}_{W_k}$*. To see that we can put scalars in front of each summand follows from an easy analysis of the argument.*

3.2. **Lie algebra realization of the compactification.** We give another realization of X, in which no choice of highest weight is used. This realization is used in [8, 9] to give a Poisson structure on the wonderful compactification. Let $n = \dim(G)$. $G \times G$ acts on $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ through the adjoint action. Let $\mathfrak{g}_\Delta = \{(x, x) : x \in \mathfrak{g}\}$, the diagonal subalgebra. Then the stabilizer in $(G \times G)$ of $\mathfrak{g}_\Delta$ is $G_\Delta = \{(g, g) : g \in G\}$, so $(G \times G) \cdot \mathfrak{g}_\Delta \cong (G \times G)/G_\Delta \cong G$.

Let $\overline{G} = \overline{(G \times G) \cdot \mathfrak{g}_\Delta}$, where the closure is computed in $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$. Since $\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ is projective, $\overline{G}$ is projective.

Proposition 3.7. $\overline{G} \cong X$*. In particular,* $\overline{G}$ *is smooth with* 2^l *orbits.*

To prove Proposition 3.7, we give a representation theoretic interpretation of $\overline{G}$, and then apply the machinery developed above.

Embed $i : \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}) \hookrightarrow \mathbb{P}(\wedge^n(\mathfrak{g} \oplus \mathfrak{g}))$ via the Plucker embedding. That is, if $U \in \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$ has basis $u_1, \dots, u_n$, we map $U \to i(U) = [u_1 \wedge \cdots \wedge u_n]$. It is well-known that the Plucker embedding is a closed embedding. Denote $[U]$ for $i(U) \in \mathbb{P}(\wedge^n(\mathfrak{g} \oplus \mathfrak{g}))$. Note that $\wedge^n(\mathfrak{g} \oplus \mathfrak{g})$ is naturally a representation of $G \times G$ by taking the exterior power of the adjoint representation.

Clearly, $i : (G \times G) \cdot \mathfrak{g}_\Delta \to (G \times G) \cdot [\mathfrak{g}_\Delta]$ is an isomorphism. Since the Plucker embedding is a closed embedding, it follows that $i : \overline{(G \times G) \cdot \mathfrak{g}_\Delta} \to \overline{(G \times G) \cdot [\mathfrak{g}_\Delta]}$ is an isomorphism.

To prove Proposition 3.7, we apply Proposition 3.5 to prove $X \cong \overline{(G \times G) \cdot [\mathfrak{g}_\Delta]}$.

Choose a nonzero vector v_Δ in the line $[\mathfrak{g}_\Delta]$, and let $E = U(\mathfrak{g} \oplus \mathfrak{g}) \cdot v_\Delta$. We show that as a $G \times G$-module, $E = \mathrm{End}(V(2\rho)) \bigoplus \oplus_i \mathrm{End}(V(\mu_i)) \bigoplus F$, where the $V(\mu_i)$ are $G \times G$-modules such that $2\rho \geq \mu_i$, and F is a $G \times G$-module. Furthermore, we show that $[\mathfrak{g}_\Delta] = [\mathrm{id}_{V(2\rho)} + \oplus_i c_i \mathrm{id}_{V(\mu_i)} + 0]$, for some scalars c_i. Then Proposition 3.5 as refined in Remark 3.6 implies the Proposition.

To verify these assertions, we first analyze the $T \times T$-weights in $\wedge^n(\mathfrak{g} \oplus \mathfrak{g})$. Let $H_1, \dots, H_l$ be a basis of $\mathfrak{t}$. A basis of $T \times T$-weights on $\mathfrak{g} \oplus \mathfrak{g}$ is given by the vectors:

(1)$\{(H_1, H_1), \dots, (H_l, H_l), (H_1, -H_1), \dots, (H_l, -H_l)\}$, all of trivial weight $(0, 0)$;

(2)$\{(E_\alpha,0):\alpha\in\Phi\}$, and note that $(E_\alpha,0)$ generates the unique weight space weight $(\alpha,0)$;

(3)$\{(0,E_\alpha):\alpha\in\Phi\}$, and note that $(0,E_\alpha)$ generates the unique weight space of weight $(0,\alpha)$.

We obtain a basis of weight vectors of $\wedge^n(\mathfrak{g}\oplus\mathfrak{g})$ as follows. Let A and B be subsets of Φ such that $|A|+|B|\le n$ and let R be a subset of the vectors in (1) of cardinality $n-|A|-|B|$. For each triple (A,B,R) as above, we define a weight vector

$$x_{A,B,R}=\wedge_{\alpha\in A}\,(E_\alpha,0)\bigwedge\wedge_{\beta\in B}\,(0,E_\beta)\bigwedge\wedge_{i=1,\dots,n-|A|-|B|}\,K_i,$$

where the K_i are the vectors in the subset R. Then the collection $\{x_{A,B,R}\}$ is a basis of weight vectors of $\wedge^n(\mathfrak{g}\oplus\mathfrak{g})$, and it is routine to check that the weight of $x_{A,B,R}$ is $(\sum_{\alpha\in A}\alpha,\sum_{\beta\in B}\beta)$. where A and B are subsets of Φ.

Take $A=\Phi^+$ and $B=-\Phi^+$ and let R_0 be the subset of basis vectors from (1) with $K_1=(H_1,H_1),\dots,K_l=(H_l,H_l)$. Then we let $v_0=x_{\Phi^+,-\Phi^+,R_0}$, so v_0 is a weight vector with weight $(2\rho,-2\rho)$. The weight of any $x_{A,B,R}$ is (λ,μ) where $\lambda\le 2\rho$ and $\mu\ge-2\rho$. It follows immediately that v_0 is a highest weight vector of $\wedge^n(\mathfrak{g}\oplus\mathfrak{g})$ of highest weight $(2\rho,-2\rho)$ relative to the Borel subgroup $B\times B^-$ of $G\times G$. Further, note that $[v_0]=[\mathfrak{t}_\Delta+\mathfrak{u}\oplus\mathfrak{u}^-]$, where $\mathfrak{t}_\Delta=\{(x,x):x\in\mathfrak{t}\}$, and $\mathfrak{u}\oplus\mathfrak{u}^-$ is the Lie algebra of the unipotent radical of $B\times B^-$.

Now we show that $v_0\in E$. Choose $H\in\mathfrak{t}$ such that $\alpha_i(H)=1$ for every simple root α_i. Define $\phi:\mathbb{C}^*\to T$ by $\phi(e^\zeta)=\exp(\zeta H)$, $\zeta\in\mathbb{C}$. We claim that:

$$\lim_{z\to\infty}(\phi(z),e)\cdot[\mathfrak{g}_\Delta]=[v_0], \tag{3.3}$$

where $z\in\mathbb{C}^*$.

To check this claim, take

$$v_\Delta=\wedge_{\alpha\in\Phi^+}(E_\alpha,E_\alpha)\bigwedge\wedge_{i=1,\dots,l}(H_i,H_i)\bigwedge\wedge_{\beta\in\Phi^+}(E_{-\beta},E_{-\beta}),$$

and note that $[v_\Delta]=[\mathfrak{g}_\Delta]$.

We decompose $(\phi(z),e)\cdot v_\Delta$ into a sum of 2^{2r} terms ($2r=|\Phi|$) in the span of weight vectors x_{A,B,R_0}, with $R_0=\{(H_1,H_1),\dots,(H_l,H_l)\}$. To get the 2^{2r} terms, for each root $\gamma\in\Phi$, decompose $(E_\gamma,E_\gamma)=(E_\gamma,0)+(0,E_\gamma)$, and choose one of the two summands in each term in the product. Each term corresponds to a choice of the subset $A\subset\Phi$, and then $B=\Phi-A$. For a root α, let $ht(\alpha)=\sum k_i$, where $\alpha=k_1\alpha_1+\cdots+k_l\alpha_l$. Then we compute

$$(\phi(z),e)\cdot x_{A,B,R_0}=\wedge_{\alpha\in A}(z^{ht(\alpha)}E_\alpha,0)\bigwedge\wedge_{i=1,\dots,l}(H_i,H_i)\bigwedge\wedge_{\beta\in B}(0,E_\beta).$$

Thus $(\phi(z), e) \cdot x_{A,B,R_0} = z^{n_A} x_{A,B,R_0}$, where $n_A = \sum_{\alpha \in A} ht(\alpha)$. Let $n_0 = \sum_{\alpha \in \Phi^+} ht(\alpha)$. It follows easily from properties of roots that $n_0 \geq n_A$ for any (A, B, R_0), and $n_0 = n_A$ if and only if $A = \Phi^+$ and $B = -\Phi^+$.

Then the formula

$$(\phi(z), e) \cdot [v_\Delta] = [z^{n_0} x_{\Phi^+, -\Phi^+, R_0} + \sum_{A \neq \Phi^+} z^{n_A} x_{A,B,R_0}]$$

implies (3.3) by taking the limit as $z \to \infty$.

It follows easily that $v_0 \in E$. Indeed, $\mathbb{P}(E)$ is $T \times T$-stable and closed, so $\overline{(T \times T) \cdot [\mathfrak{g}_\Delta]} \subset \mathbb{P}(E)$. It follows from (3.3) that $[v_0] \in \mathbb{P}(E)$, so $v_0 \in E$.

Now observe that since v_0 is a highest weight vector of weight $(2\rho, -2\rho)$, $U(\mathfrak{g} \oplus \mathfrak{g}) \cdot v_0 \cong V(2\rho) \otimes V(-2\rho) \cong \mathrm{End}(V(2\rho))$ is a submodule of E. Moreover, by our determination of weights of $\wedge^n(\mathfrak{g} \oplus \mathfrak{g})$, any highest weight vector occurring in E has highest weight $\mu \leq 2\rho$. It follows by separating out the irreducible $G \times G$ representations in E of the form $\mathrm{End}(V(\mu))$ that

$$E = \mathrm{End}(V(2\rho)) \bigoplus \oplus_i \mathrm{End}(V(\mu_i)) \bigoplus F, \tag{3.4}$$

where F is a sum of irreducible representation of $G \times G$ not isomorphic to $\mathrm{End}(V(\mu))$ for any μ.

It remains to verify the claim that $[\mathfrak{g}_\Delta] = [\mathrm{id}_{V(2\rho)} + \sum c_\mu \mathrm{id}_{V(\mu)}]$. Since v_Δ is G_Δ-invariant, it follows that its projection to each irreducible $(G \times G)$-representation appearing in (3.4) is G_Δ-invariant. We recall the classification of irreducible representations with a nonzero G_Δ-fixed vector.

Lemma 3.8. *Let E be an irreducible representation of $G \times G$. Then E has a nonzero G_Δ-fixed vector if and only if $E \cong \mathrm{End}(W)$ for some irreducible representation W of G. If $E = \mathrm{End}(W)$, then its G_Δ-fixed space $\mathbb{C} \cdot \mathrm{id}_W$ is one dimensional.*

Proof. Every irreducible representation of $G \times G$ is isomorphic to $V \otimes W^*$, where V and W are irreducible represenations of G. Schur's Lemma implies that $V \otimes W^* \cong \mathrm{Hom}(W, V)$ has a nonzero G_Δ-invariant vector if and only $W \cong V$, so $V \otimes W^* \cong \mathrm{End}(V)$. Moreover, by Schur's Lemma, if $W = V$, the space of invariants is generated by the identity.

Q.E.D.

In particular, by this Lemma, the projection of v_Δ to each factor in (3.4) is a scalar times the identity. Thus, we may write:

$[\mathfrak{g}_\Delta] = [c_\rho \mathrm{id}_{V(2\rho)} + \sum c_\mu \mathrm{id}_{V(\mu)}]$.

To complete the proof, it suffices to show that c_ρ is nonzero. For this, it suffices to show the projection of v_Δ to $\mathrm{End}(V(2\rho))$ in (3.4) is nonzero. We verified in the proof of (3.3) that when we write v_Δ as a sum of our standard basis vectors, v_0 is a nonzero summand of weight $(2\rho, -2\rho)$. The fact that the projection is nonzero follows using the $T \times T$-action and linear independence of distinct eigenvalues.

Q.E.D.

Remark 3.9. *There is a unique $G \times G$-isomorphism $\phi : X \to \overline{G}$ such that $[\mathrm{id}_V] \mapsto \mathfrak{g}_\Delta$. We may use ϕ to find representatives for each of the $G \times G$-orbits in $\overline{G}$.*

The points z_I in X correspond to Lie subalgebras $\phi(z_I) = \mathfrak{m}_I$ of $\mathfrak{g} \oplus \mathfrak{g}$. Recall $\mathfrak{l}_I$, $\mathfrak{u}_I$ and $\mathfrak{u}_I^-$ from section 2.5. Then,

$$\mathfrak{m}_I = \{(X + Z, Y + Z) : X \in \mathfrak{u}_I, Y \in \mathfrak{u}_I^-, Z \in \mathfrak{l}_I\}.$$

This may verified as follows. Find a curve $f : \mathbb{P}^1 \to X$ with image in the toric variety $X' = \overline{(T \times \{e\}) \cdot [\mathrm{id}_V]}$ such that $f(0) = \mathrm{id}_V$ and $f(\infty) = z_I$. Using $T \times T$-equivariance, consider the image of this curve in $\overline{G}$ and $\phi \circ f(\infty) = \phi(z_I)$. The morphism ϕ is completely determined by the $\phi(z_I)$ and $G \times G$-equivariance, and from this it is not difficult to show that $\phi(z_I) = \mathfrak{m}_I$.

These subalgebras at infinity are used in the paper [8] *to compute real points of the wonderful compactifiction for a particular real form.*

4. Cohomology of the compactification

4.1. **$T \times T$-fixed points on X.** We explain how to compute the integral cohomology of X following [6]. We use the Bialynicki-Birula decomposition. Suppose that Z is any smooth projective variety with a $\mathbb{C}^*$-action and suppose the fixed point set $Z^{\mathbb{C}^*}$ is finite. The idea is that we should be able to recover the topology of Z from the fixed point set $Z^{\mathbb{C}^*}$ and the action of $\mathbb{C}^*$ on the tangent space at fixed points.

We make some remarks about the action. Suppose $m = \dim(Z)$. Let $z_0 \in Z$ be a fixed point for the $\mathbb{C}^*$-action. Then $\mathbb{C}^*$ acts linearly on the tangent space $T_{z_0}(Z)$. It follows that $T_{z_0}(Z) = \sum_{j=1}^m \mathbb{C}v_j$ where $a \cdot v_j = a^{n_j} v_j$. Then each n_j is nonzero, and the weights $(n_1, \ldots, n_m)$ are called the weights of $\mathbb{C}^*$ at z_0.

Let $T_{z_0}^+(Z) = \sum_{n_j > 0} \mathbb{C}v_j$.

Theorem 4.1. *(Bialynicki-Birula* [2]*)*
Let Z be as above, and let $Z^{\mathbb{C}^*} = \{z_1, \ldots, z_n\}$. For $i \in \{1, \ldots, n\}$, let
$C_i = \{z \in Z : \lim_{a \to 0} a \cdot z = z_i\}$. Then

1. $Z = \cup_i C_i$
2. $C_{z_i} \cong T_{z_i}^+(Z)$
3. C_{z_i} *is locally closed in Z.*

As a consequence, $H_*(Z) = \sum_{i=1}^n \mathbf{Z}\sigma_i$, where $\sigma_i \in H_{2\dim(C_{z_i})}(Z)$ is the cycle corresponding to $\overline{C_{z_i}}$.

For example, if $X = \mathbb{P}(\mathbb{C}^{n+1})$, we may let $\mathbb{C}^*$ act on X by

$$a \cdot [(x_0, \ldots, x_n)] = [(a^n x_0, a^{n-1} x_1, \ldots, a x_{n-1}, x_n)], a \in \mathbb{C}^*$$

Then the fixed points are the coordinate vectors and the corresponding decomposition from Theorem 4.1 is the usual cell decomposition of $\mathbb{P}(\mathbb{C}^{n+1})$.

We want to do a similar analysis for $X = \overline{\psi(G)}$. We will find a subgroup of $T \times T$ isomorphic to $\mathbb{C}^*$, prove that $X^{T\times T} = X^{\mathbb{C}^*}$, and compute the Bialynicki-Birula decomposition. To do this, we need some results on $X^{T\times T}$ and the action of $T \times T$ on the tangent space to the fixed points.

For $w \in W$, choose a representative $\dot{w} \in N_G(T)$, and for notational simplicity we use the identity of G as a representative for the identity of W. For simplicity, denote z_0 for $z_{\{1,\ldots,l\}}$. We remark that the unique closed orbit $(G\times G)\cdot z_0$ is isomorphic to $G/B\times G/B^-$.

Lemma 4.2. $X^{T\times T} = \cup_{(y,w)\in W\times W}\ z_{y,w}$*, where* $z_{y,w} = (\dot{y}, \dot{w}) \cdot z_0$*.*

Proof. We first claim that $X^{T\times T} \subset (G \times G) \cdot z_0$. Since $X = \cup_I\ (G \times G) \cdot z_I$, it suffices to consider any point $x \in (G \times G) \cdot z_I$. If $x \in X^{T\times T}$, then $(G \times G)_x$ must contain a torus of dimension $2l$. But the stabilizer $(G \times G)_x$ is isomorphic to the stabilizer $(G \times G)_{z_I}$. By the computation in Proposition 2.25, a maximal torus of $(G \times G)_{z_I}$ is the set

$\{(t_1, t_2) \in T \times T : t_1 {t_2}^{-1} \in Z(L_I)\}$.

It is easy to see that this maximal torus has dimension $l + |I|$. In particular, $(G \times G)_{z_I}$ contains a $2l$-dimensional torus if and only if $I = \{1, \ldots, l\}$. The claim follows.

Recall the well-known fact that $(G/B)^T = \{\dot{w}B : w \in W\}$. The Lemma now follows by using the isomorphism $(G \times G) \cdot z_0 \cong G/B \times G/B^-$.

Q.E.D.

Lemma 4.3. *Let* $z_0 = z_{\{1,\ldots,l\}}$*. The weights of* $T \times T$ *on the tangent space* $T_{z_0}(X)$ *are*

(1) $(\frac{1}{\alpha}, 1) : \alpha \in \Phi^+$;

(2) $(1, \alpha) : \alpha \in \Phi^+$;

(3) $(\frac{1}{\alpha_i}, \alpha_i) : i = 1, \ldots, l$

Proof. Since X_0 is an open neighborhood of z_0 in X, the tangent space $T_{z_0}(X) = T_{z_0}(X_0)$. Since $X_0 \cong U^- \times U \times Z$,

$$T_{z_0}(X_0) \cong \mathfrak{u}^- \oplus \mathfrak{u} \oplus T_{z_0}(Z).$$

The first factor of T acts on $\mathfrak{u}^-$ by the adjoint action and the second factor acts trivially, and this gives the weights appearing in (1). The first factor of T acts trivially on $\mathfrak{u}$ and the second factor acts by the adjoint action, which gives the weights appearing in (2). We define a $T \times T$-action on $\mathbb{C}^l$ by

$$(t_1, t_2) \cdot (z_1, \ldots, z_l) = (\alpha_1(t_2/t_1)z_1, \ldots, \alpha_l(t_2/t_1)z_l), t_1, t_2 \in T.$$

We claim that $F : \mathbb{C}^l \to Z$ is $T \times T$-equivariant. The weights in (3) are easily derived from this claim. The claim for $T \times \{e\}$ follows the same way as the derivation of the final formula for $\psi(t)$ in the proof of Lemma 2.7. The claim for $\{e\} \times T$ follows by computing $(e, t) \cdot \psi(e)$ by the same method as in 2.7. The answer is different because $(e, t) \cdot v_k \otimes v_k^* = v_k \otimes \frac{1}{\lambda_k(t)} v_k^*$, which is the inverse of the corresponding formula for the (t, e)-action. We leave the details to the reader.

Q.E.D.

Let $y, w \in W$. Define $\phi : X \to X$ by $\phi(x) = (\dot{y}, \dot{w}) \cdot x$. Then $z_{y,w} = \phi(z_0)$ is the $T \times T$-fixed point of X corresponding to (yB, wB^-).

Lemma 4.4. *The weights of $T \times T$ on the tangent space $T_{z_{y,w}}(X)$ are*

(1) $(\frac{1}{y\alpha}, 1) : \alpha \in \Phi^+$;

(2) $(1, w\alpha) : \alpha \in \Phi^+$;

(3) $(\frac{1}{y\alpha_i}, w\alpha_i) : i = 1, \ldots, l$

Proof. Indeed, let $(t_1, t_2) \in T \times T$. It is routine to check that

$$\phi((t_1, t_2) \cdot x) = (y(t_1), w(t_2))\phi(x).$$

Thus, the differential $\phi_* : T_{z_0}(X) \to T_{z_{y,w}}(X)$ satisfies the formula:

$$\phi_*(t_1, t_2) = (y(t_1), w(t_2))\phi_*.$$

In particular, if $\lambda_1, \ldots, \lambda_n$ are the weights of $T \times T$ at z_0, then $(y, w)\lambda_1, \ldots, (y, w)\lambda_n$ are the $T \times T$-weights at $z_{y,w}$. The result now follows from Lemma 4.3.

Q.E.D.

Remark 4.5. *By Lemma 4.3, it follows that the $\{e\} \times T$ weights at the origin in Z are $\alpha_i, i = 1, \ldots, l$. By the proof of Lemma 4.4, it follows that the $\{e\} \times T$ weights at the point $z_{w,w}$ are $w(\alpha_i), i = 1, \ldots, l$. Using these weight calculations, we may identify $\tilde{Z} := \cup_{w \in W} (\dot{w}, \dot{w}) \cdot Z$ with the closure of Z in X. Indeed, since Z is smooth, it follows easily that $\tilde{Z}$ is a smooth toric variety for the torus $\{e\} \times T \cong T$. Every toric variety Y with torus S has a fan, which is a subset of $\mathfrak{s}_{\mathbb{R}} = X_*(S) \otimes_{\mathbf{Z}} \mathbb{R}$. To compute the fan of a smooth toric variety, we compute the weights $\lambda_1, \ldots, \lambda_l$ of the action of the torus S at each fixed point z. Define the chamber C_z for z by $C_z = \{x \in \mathfrak{s}_{\mathbb{R}} : \lambda_i(x) \geq 0, i = 1, \ldots, l\}$. Then the fan of Y is the subset $F(Y) := \cup_{z \in Y^S} C_z$ together with the origin. The fan is called complete if $F(Y) = \mathfrak{s}_{\mathbb{R}}$, and if $F(Y)$ is complete, then the toric variety Y is complete. For $\tilde{Z}$, the fan is easily identified with the Weyl chamber decomposition of $\mathfrak{t}_{\mathbb{R}}$, and in particular, it is complete. It is easy to see that Z is dense in $\tilde{Z}$, and from this it follows that $\tilde{Z}$ coincides with the closure of Z in X. See* [10] *for basic facts about toric varieties.*

4.2. **Computation of cohomology.** Recall the following well-known fact.

Lemma 4.6. *Let Y be a smooth $\mathbb{C}^*$-variety. Let Y' be a connected component of $Y^{\mathbb{C}^*}$.*

(1) Y' is smooth and projective.

(2) Y' is a point if and only if there is a point $y \in Y'$ such that all the weights of $\mathbb{C}^*$ on $T_y(Y)$ are nontrivial.

The condition that $\mathbb{C}^*$ has no trivial weights on $T_y(Y)$ is equivalent to the condition that the differentiated representation of the Lie algebra $\mathbb{C}$ of $\mathbb{C}^*$ on $T_y(Y)$ has no zero weights. It follows that to find a $\mathbb{C}^*$-action on X with no fixed points, it suffices to find some $A \in \mathfrak{t} \oplus \mathfrak{t}$ with no zero weights on $T_{z_{y,w}}(X)$ for all pairs $(y, w) \in W \times W$. The element A generates $Lie(\mathbb{C}^*) = \mathbb{C} \cdot A$. Write $d\alpha \in \mathfrak{t}^*$ for the differential of α.

Choose $H \in \mathfrak{t}$ so that $d\alpha_i(H) = 1$ for all simple roots α_i. It follows that if α is any root, $|d\alpha(H)| \geq 1$. We may choose n sufficiently large so that $n > d\beta(H)$ for every root β. Then $|nd\alpha(H)| > |\beta(H)|$ for all roots α and β. In particular,

$$nd\alpha(H) > 0 \iff nd\alpha(H) + \beta(H) > 0 \tag{4.1}$$

for every pair of roots α and β.

Consider $\eta : \mathbb{C}^* \to T \times T$ given by $\eta(a) = (\phi(a^n), \phi(a^{-1}))$. Then $d\eta(1) = (nH, -H)$.

We will let $\mathbb{C}^*$ act on the wonderful compactification X via η, so $a \cdot x = \eta(a) \cdot x$.

Lemma 4.7. $X^{T\times T} = X^{\mathbb{C}^*}$.

Proof. Indeed, it is clear that $X^{T\times T} \subset X^{\mathbb{C}^*}$. Let X' be a connected component of $X^{\mathbb{C}^*}$. Then X' is projective by Lemma 4.6 (1), and is $T \times T$-stable since $T \times T$ commutes with $\mathbb{C}^*$. The Borel fixed point theorem asserts that an action of a connected solvable group on a projective variety has a fixed point. Thus, X' contains a $T \times T$-fixed point. By Lemma 4.2, some $z_{y,w} \in X'$.

We compute the weights of $(nH, -H)$ on the tangent space $T_{z_{y,w}}(X)$ and show they are all nonzero. It follows that all weights of $(nH, -H)$ on $T_{z_{y,w}}(X')$ are nonzero. Then Lemma 4.6 (2) implies our claim.

By Lemma 4.4, the eigenvalues of $(nH, -H)$ on $T_{z_{y,w}}(X)$ are:

(1) $-ndy\alpha(H), \alpha \in \Phi^+$

(2) $-dw\alpha(H), \alpha \in \Phi^+$

(3) $-(ndy\alpha_i(H) + dw\alpha_i(H)), \alpha_i$ simple.

By the choice of H, the numbers in (1) and (2) are nonzero. By (4.1) above, the numbers in (3) are nonzero.

Q.E.D.

For $y \in W$, let

$$L(y) = |\{\alpha_i \in \Delta : y\alpha_i \in -\Phi_+\}|.$$

Theorem 4.8.
$$H^*(X) = \sum_{(y,w)\in W\times W} \mathbf{Z}\sigma_{y,w},$$
where $\sigma_{y,w}$ has degree $2(l(y)+l(w)+L(y))$.

Proof. For the $\mathbb{C}^*$-action on X from Lemma 4.7, $X^{\mathbb{C}^*} = \{z_{y,w} : (y,w) \in W \times W\}$. By Theorem 4.1 and the remarks following Lemma 4.6, it suffices to compute the number of positive eigenvalues of the generator $(nH, -H)$ of the Lie algebra of $\mathbb{C}^*$ on the tangent space at each fixed point.

By the definition of H, $-dy\alpha(H) > 0$ if and only if $y(\alpha) \in -\Phi^+$. By the definition of n, $-(ndy\alpha_i(H) + dw\alpha_i(H)) > 0$ if and only if $y\alpha_i \in -\Phi^+$. By the computation at the end of the proof of Lemma 4.7, it follows that the number of positive eigenvalues of the generator $(nH, -H)$ at the tangent space at $z_{y,w}$ is $l(y) + l(w) + L(y)$ (here we use the well-known fact that $l(y)$ is the number of positive roots whose sign is changed by y). The Theorem follows.

Q.E.D.

Remark 4.9. *The asymmetry of the roles of y and w in the statement of the Theorem is due to the need to choose a $\mathbb{C}^*$-action without fixed points.*

5. Appendix on compactifications of general symmetric spaces

Let G be complex semisimple with trivial center, and let $\sigma : G \to G$ be an algebraic involution with fixed subgroup $H = G^\sigma = \{x \in G : \sigma(x) = x\}$. Then we call (G,H) a symmetric pair and the associated homogeneous space G/H is called a semisimple symmetric space. As a special case, let G_1 be complex semisimple with trivial center and let $G = G_1 \times G_1$. Then the diagonal subgroup $G_{1,\Delta}$ is the fixed point set of the involution $\sigma : G \to G$ given by $\sigma(x,y) = (y,x)$. We refer to the symmetric pair $(G_1 \times G_1, G_{1,\Delta})$ as the group case, and note that the map $G_1 \cong (G_1 \times G_1)/G_{1,\Delta} \to G_1$, $(x,y) \mapsto xy^{-1}$ identifies the group G_1 as a symmetric space. The purpose of this appendix is to explain a construction of the DeConcini-Procesi compactification of the semisimple symmetric space G/H. In the group case, this construction is not identical to the construction given in Section 2, but it is equivalent.

First, we need some structure theory. For a σ-stable maximal torus T, let $T^{-\sigma} = \{t \in T : \sigma(t) = t^{-1}\}$. T is called maximally split if $\dim(T^{-\sigma})$ is maximal among all σ-stable maximal tori of G. All maximally split tori are conjugate in G [17]. For T maximally split, let A be the connected component of the identity of $T^{-\sigma}$ and let $r = \dim(A)$. Let $\mathfrak{a}$ be the Lie algebra of A. We call r the split rank of the pair (G,H). In the group case, if T_1 is a maximal torus of G_1, $T_1 \times T_1$ is σ-stable and $A = \{(t, t^{-1}) : t \in T_1\}$, so the split rank is $\dim(T_1)$, which is the rank of G.

Let T be maximally split. Since T is σ-stable, there is an induced involution on the root system Φ of (G,T), given by $\sigma(\alpha)(t) = \alpha(\sigma^{-1}(t))$. Then $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}$. We say a root α is imaginary if $\sigma(\alpha)=\alpha$. There exists a positive system Φ^+ such that if $\alpha\in\Phi^+$ and $\sigma(\alpha)\in\Phi^+$, then α is imaginary (see [DCP], Lemma 1.2). Let $S=\{\beta_1,\ldots,\beta_s\}$ be the simple imaginary roots for Φ^+, and let $[S]=\{\beta\in\Phi:\beta=\sum n_j\beta_j\}$. Clearly, σ acts as the identity on $[S]$.

CLAIM: If β is an imaginary root, $\beta\in[S]$.

To prove the claim, let $\alpha_1,\ldots,\alpha_t$ be the simple nonimaginary roots and define $f\in\mathfrak{t}^*=X(T)^*\otimes\mathbb{C}$ so that $f(\alpha_i)=1$ if α_i is a simple non-imaginary root, and $f(\beta_j)=0$ if β_j is imaginary. Note that if α is a root, then

(1)$f(\alpha)>0 \iff \alpha\in\Phi^+-[S]$,

(2)$f(\alpha)=0 \iff \alpha\in[S]$,

(3)$f(\alpha)<0 \iff \alpha\in-\Phi^+-[S]$.

Suppose β is positive imaginary and write $\beta=\sum_{n_i\geq 0}n_i\alpha_i+\sum_{m_j\geq 0}m_j\beta_j$, where the first sum is over simple nonimaginary roots and the second sum is over simple imaginary roots. Then $\beta=\sigma(\beta)=\sum n_i\sigma(\alpha_i)+\sum m_j\beta_j$. Since α_i is not imaginary, $\sigma(\alpha_i)\in-\Phi^+$. Moreover, $\alpha_i\notin[S]$, so $\sigma(\alpha_i)\notin[S]$. It follows that $f(\sigma(\alpha_i))<0$, and hence that $f(\sigma(\beta))<0$ if some $n_i\neq 0$. But $f(\beta)\geq 0$, so $f(\beta)=0$ and all $n_i=0$, so $\beta\in[S]$.

Let $\mathfrak{m}=\mathfrak{t}+\sum_{\alpha\in[S]}\mathfrak{g}_\alpha$. Then $\mathfrak{m}$ is a Levi subalgebra whose roots form the full set of imaginary roots. Moreover, $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$. Let $\mathfrak{p}=\mathfrak{m}+\mathfrak{u}$, where $\mathfrak{u}$ is the unipotent radical of the Borel subalgebra determined by Φ^+. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{p}$, and let $\mathfrak{n}^-$ be the opposite T-stable nilradical, so $\mathfrak{g}=\mathfrak{p}+\mathfrak{n}^-$. Then $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is the sum of the positive (resp. negative) nonimaginary root spaces. Let P,M,N and N^- be the corresponding connected groups. If $\mathfrak{g}_\mathbb{R}$ is the real form of $\mathfrak{g}$ such that $\mathfrak{h}$ is the complexification of a maximal compact subalgebra of $\mathfrak{g}_\mathbb{R}$, then $\mathfrak{p}$ is the complexification of a minimal parabolic subalgebra of $\mathfrak{g}_\mathbb{R}$, and $\mathfrak{a}$ and $\mathfrak{n}$ are complexifications of corresponding factors in an Iwasawa decomposition of $\mathfrak{g}_\mathbb{R}$.

Let α_i be a non-imaginary simple root. Then $\sigma(\alpha_i)=-\alpha_j-\sum_{i=1}^s k_i\beta_i$ for some nonimaginary simple root α_j and some nonnegative integers k_i (see [DCP], Lemma 1.4). In this case, we define $\sigma(i)=j$, and note that $\sigma(j)=i$. σ defines an involution on the indexing set of nonimaginary simple roots $\{1,\ldots,t\}$ with r orbits. Renumber the $\alpha_1,\ldots,\alpha_t$ so that each orbit of σ on $\{1,\ldots,t\}$ has one element in $\{1,\ldots,r\}$. If β is imaginary, then $d\beta$ vanishes on $\mathfrak{a}$, and $\sigma(d\alpha_i)(Y)=-d\alpha_j(Y)$ for all $Y\in\mathfrak{a}$. By definition, the restricted roots are the set of nontrivial characters $\{\alpha|_A\colon \alpha\in\Phi\}$. For α non-imaginary, set $\overline{\alpha}=\alpha|_A$. We call $\overline{\alpha_1},\ldots,\overline{\alpha_r}$ the simple restricted roots. Then each restricted root is a sum of simple restricted roots. We may use the direct sum decomposition $\mathfrak{t}=\mathfrak{t}^\sigma+\mathfrak{a}$ to identify $\mathfrak{a}^*\cong\{f\in\mathfrak{t}^*: f(\mathfrak{t}^\sigma)=0\}$. For α nonimaginary, $d\overline{\alpha}$ corresponds to $\frac{d\alpha-\sigma(d\alpha)}{2}$ in this identification. If $\sigma(\alpha_i)=-\alpha_i$, we call α_i a real root, and if $\sigma(\alpha_i)\neq\pm\alpha_i$, we call α_i a complex root. The Satake diagram of $(\mathfrak{g},\sigma)$ is the Dynkin diagram of $\mathfrak{g}$ with simple imaginary roots colored black, and other simple roots colored white, with a double-edged arrow connecting α_i and α_j if $\sigma(i)=j$. The Satake diagram determines σ up to isomorphism [1].

Let $\tilde{H}$ be the preimage of H in the simply connected cover $\tilde{G}$ of G. Let V be an irreducible representation of $\tilde{G}$ with highest weight vector v_0 of weight λ. The induced $\tilde{G}$-action on $\mathbb{P}(V)$ factors to give a G-action on $\mathbb{P}(V)$ and similarly, we obtain a $G \times G$-action on $\mathbb{P}(\mathrm{End}(V))$. We choose λ so that $G_{[v_0]} = P$, where $G_{[v_0]}$ is the stabilizer in G of the line $[v_0]$ through v_0. Equivalently, $\lambda(\alpha_i) > 0$ for every simple nonimaginary root α_i, and $\lambda(\beta_i) = 0$ for every simple imaginary root β_i. In [5], such a weight λ is called *regular special*. Complete v_0 to a basis $v_0, \dots, v_n$ of weight vectors, and number the weights so v_i has weight $\lambda - \alpha_i$, for $i = 1, \dots, r$. Choose also a dual basis $v_0^*, v_1^*, \dots, v_n^*$. Let $\mathbb{P}_0(V) = \{[v] \in \mathbb{P}(V) : v_0^*(v) \neq 0\}$. The hypothesis on λ implies that the morphism $N^- \to N^- \cdot [v_0]$ is an isomorphism, and the image is closed in $\mathbb{P}_0(V)$ by Lemma 2.4.

Consider the twisted conjugation action of G on $\mathbb{P}(\mathrm{End}(V))$ given by $g \cdot [C] = g[C]\sigma(g^{-1})$ for $g \in G$ and $C \in \mathrm{End}(V)$. Let $h = \mathrm{id}_V$. Then the stabilizer $G_{[h]}$ of the line $[h]$ is H, so the orbit $G \cdot [h] \cong G/H$, and we may regard the irreducible subvariety $X := \overline{G \cdot [h]}$ as a compactification of G/H. We wish to show X is smooth and describe its geometry.

For this, let $\mathbb{P}_0 = \{[C] \in \mathbb{P} : v_0^*(C \cdot v_0) \neq 0\}$ and let $X_0 = X \cap \mathbb{P}_0$. As in 2.2, we identify X_0 with an affine space, and then show X is a union of translates of X_0. For this, note that the identification $V \otimes V^* \cong \mathrm{End}(V)$ is G-equivariant, where we use the twisted conjugation action on $\mathrm{End}(V)$ and the G-action on $V \otimes V^*$ is the unique action such that $g \cdot (v \otimes f) = g \cdot v \otimes \sigma(g) \cdot f$ (the G-action on V^* is the usual action on the dual). We now compute the A-action on $[h]$. We identify $\mathrm{id}_V = \sum v_i \otimes v_i^*$ and obtain

$$t \cdot [h] = t \cdot [\sum v_i \otimes v_i^*] = [v_0 \otimes v_0^* + \sum_{i=1}^{n} \frac{1}{(\gamma_i - \sigma(\gamma_i))(t)} v_i \otimes v_i^*],$$

where $\lambda - \gamma_i$ is the weight of v_i, by reasoning as in 2.7. For $j > 0$, $\gamma_j - \sigma(\gamma_j) = \sum_{i=1}^{r} n_{ij}(\alpha_i - \sigma(\alpha_i))$ for nonnegative integers n_{ij}. It follows that the morphism $\psi : \mathbb{C}^r \to \mathbb{P}_0$ given by

$$(z_1, \dots, z_r) \mapsto [v_0 \otimes v_0^* + \sum_{i=1}^{r} z_i v_i \otimes v_i^* + \sum_{j=r+1}^{n} \prod_{i=1}^{r} {z_i}^{n_{ij}} v_j \otimes v_j^*]$$

is an isomorphism. Let $Z = \psi(\mathbb{C}^r)$. Note that A acts on Z through its quotient A/D, where $D = \{a \in A : \alpha_i(a) = \pm 1, i = 1, \dots, r\}$.

Note that N^- preserves $\mathbb{P}_0$ so N^- acts on X_0. Indeed, if $v_0^*(C \cdot v_0) \neq 0$, then for $n \in N^-$, $v_0^*(n \circ C \circ \sigma(n)^{-1} \cdot v_0) = v_0^*(n \cdot C \cdot v_0) = v_0^*(C \cdot v_0)$ since $\sigma(n) \in N$ and n fixes the lowest weight vector v_0^* of V^*. Thus, we have the morphism $\chi : N^- \times \mathbb{C}^r \to X_0$ given by $\chi(u, z) = u \cdot \psi(z)$. We claim that χ is an isomorphism. For this, by reasoning as in the proof of 2.8, it suffices to construct a N^--equivariant morphism $\beta : X_0 \to N^-$ such that $\beta \circ \chi(u, z) = u$. To construct β, consider the map $\nu : \mathbb{P}_0 \to \mathbb{P}_0(V)$ defined by $\nu([C]) = [C \cdot v_0]$. Note that

$$\nu(N^- A \cdot [h]) = \{\nu([x\sigma(x)^{-1}]) : x \in N^- A\} =$$

$$\{[x\sigma(x)^{-1} \cdot v_0] : x \in N^- A\} = \{[x \cdot v_0] : x \in N^- A\} = N^- \cdot [v_0].$$

It is easy to check that $\dim(N^- A \cdot [h]) = \dim G \cdot [h]$, and it follows that $N^- A \cdot [h]$ is open and dense in X_0, since an orbit of an algebraic group on a variety is open in its closure. It follows that $\nu(X_0) \subset N^- \cdot [v_0]$, since $N^- \cdot [v_0]$ is closed in $\mathbb{P}_0(V)$. We thus obtain a morphism $\beta : X_0 \to N^-$ by composing ν with the isomorphism $N^- \cdot [v_0] \to N^-$. It is easy to check that β has the required property, and this gives the following theorem.

Theorem 5.1. *$\chi : N^- \times \mathbb{C}^r \to X_0$ is an isomorphism of varieties. In particular, X_0 is smooth.*

To prove X is smooth, let $V = \cup_{g \in G}\, g \cdot X_0$. We want to show $V = X$. For this, the following result from [7] is useful.

Proposition 5.2. *Let X be a complex projective G-variety. Let $x \in X$ be a point with stabilizer $G_x = H$. Let $Y \subset X$ be a locally closed G-stable subset containing the orbit $G \cdot x$ and suppose that*

(a) $G \cdot x$ is dense in Y

(b) The closure of $T \cdot x$ in Y is projective.

Then Y is projective.

The proof, given in [7], uses the valuative criterion for properness and a variant of the Hilbert-Mumford criterion.

To apply this result to show $V = X$, it suffices to show that the closure X' of $T \cdot [h]$ in V is complete. For this, we use the theory of toric varieties. Let $W_A = N_G(A)/Z_G(A)$, the so-called little Weyl group. For $w \in W_A$, let $\dot{w}$ be a representative in $N_G(A)$. Clearly $Z \subset X'$, and it follows that $\cup_{w \in W_A}\, \dot{w} \cdot Z$ is contained in X'. But $\cup_{w \in W_A}\, \dot{w} \cdot Z$ is a smooth toric variety for the torus A/D, and it is straightforward as in Remark 4.5 to show that its fan is the Weyl chamber decomposition of $\mathfrak{a}_{\mathbb{R}}$. In particular, the fan is complete, so $\cup_{w \in W_A}\, \dot{w} \cdot Z$ is a complete variety by standard results on toric varieties [10]. Since $\cup_{w \in W_A}\, \dot{w} \cdot Z$ is dense in X', it folows easily that X' is complete, so it is projective. Now Proposition 5.2 implies that V is projective, so $V = X$.

Further, let $Z_i = \psi(\{(z_1, \ldots, z_r) : z_i \neq 0\}$, and let $D_i = \overline{G \cdot Z_i}$. Then by arguments as in 2.4, each D_i is a smooth divisor, and the different D_i meet transversally. For $I \subset \{1, \ldots, r\}$, let $D_I = \cap_{i \in I} D_i$.

Theorem 5.3. *The following hold:*

1. *X is smooth.*
2. *Every G-orbit closure in X is D_I for some I. In particular, all G-orbit closures are smooth.*
3. *For each G-orbit closure D_I in X, there is a parabolic subgroup Q_I such that D_I fibers over G/Q_I with fiber a wonderful compactification of a symmetric space of the adjoint quotient of the Levi factor of Q_I.*

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Department of Mathematics, University of Notre Dame, Notre Dame, 46556

E-mail address: sevens@nd.edu

Department of Mathematics, University of Georgia, Athens, GA 30602-3704

E-mail address: bjones@math.uga.edu