Transverse measures, the modular class, and a cohomology pairing for Lie algebroids

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Abstract

We show that every Lie algebroid A over a manifold P has a natural representation on the line bundle $Q_A = \wedge^{top} A \otimes \wedge^{top} T^*P$. Sections of Q_A may be viewed as transverse measures to A. The obstruction to the existence of an A-invariant section of Q_A defines a class in the Lie algebroid cohomology $H^1(A)$, called the modular class of the Lie algebroid A. (It was introduced earlier by Weinstein using the Poisson structure on A^*.) A natural pairing between the Lie algebroid cohomology spaces of A with trivial coefficients and with coefficients in Q_A generalizes the standard pairings for de Rham cohomology and Lie algebra cohomology. The modular class is interpreted in terms of an "adjoint representation up to homotopy" for a Lie algebroid, and a similar construction is given for Lie groupoids.

1 Introduction

The notion of Lie algebroids [Ma] generalizes that of both Lie algebras and tangent bundles. Correspondingly, representations of Lie algebroids are generalizations of Lie algebra representations and vector bundles with flat connections.

In this paper, we construct, for an arbitrary Lie algebroid A over a manifold P, a representation of A on the line bundle $Q_A = \wedge^{top} A \otimes \wedge^{top} T^*P$. (The notation $\wedge^{top}$ refers to the highest exterior power.) When A is the subbundle of TP tangent to a foliation $\mathcal{F}$, sections of Q_A are the transverse measures to $\mathcal{F}$, and we recover the top exterior power of

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the Bott connection for this foliation. (The Bott connection itself is generalized to the case of an arbitrary Lie algebroid in Appendix A.) For this reason, we may think of sections of Q_A as transverse measures to any Lie algebroid A.

We give two applications of this construction.

The first application concerns the modular class of A. Every representation of A on a line bundle defines a "characteristic class" in the first Lie algebroid cohomology of A (with trivial coefficients). When this is applied to the representation on Q_A, we get the modular class of A which was first defined in [We] by using the Poisson structure on the dual bundle A^*. When A is the cotangent bundle Lie algebroid T^*P of a Poisson manifold P, using the representation of A on $Q_A = (\wedge^{top}T^*P)^2$, we in fact get the representation of A on the "square root" $\wedge^{top}T^*P$ of Q_A. The corresponding characteristic class of A is then the modular class of the Poisson structure defined in [We]. We also show that Poisson homology studied by Koszul [Kz] and Brylinski [Br] is isomorphic to the Lie algebroid cohomology of $A = T^*P$ with coefficients in $\wedge^{top}T^*P$.

As the second application, we establish a pairing between the Lie algebroid cohomology spaces of A with trivial coefficients and with coefficients in Q_A. This generalizes the standard pairings that give Poincaré duality for Lie algebra cohomology and de Rham cohomology. For Poisson manifolds, we get a pairing on the Poisson homology. These pairings may be degenerate in general, as is seen by examples. The problem of when it is non-degenerate is very interesting. A detailed study of Poincaré duality for Lie-Rinehart algebras, an algebraic generalization of Lie algebroids, is given in [H].

Although there is no natural "adjoint" representation of a general Lie algebroid A (on itself), we show in Appendix A that there is a representation *up to homotopy* of A on the "formal difference" $A \ominus TP$. Taking its top exterior power yields the representation on Q_A, thus providing another construction for this representation. In Appendix B, we discuss the modular class of a Lie groupoid.

Throughout this paper, we will denote by $\Gamma(E)$ the space of smooth sections of a vector bundle E, with a few exceptions. The space of k-vector fields on a manifold P, i.e., smooth sections of $\wedge^k TP$, will be denoted by $\chi^k(P)$ and the space of differential k-forms on P by $\Omega^k(P)$. If E is a vector bundle and if $X \in \Gamma(\wedge^k E)$, we will use i_X or $X \lrcorner$ to denote the operation on $\Gamma(\wedge E^*)$ given by contraction with X.

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2 Differential calculus on Lie algebroids

In this section, we recall some facts on the calculus on Lie algebroids that will be used in this paper. See [Ma] and [Va] for more details.

A *Lie algebroid* over a smooth manifold P is a vector bundle A over P together with 1) a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A, and 2) a bundle map $\rho : A \to TP$, such that

1) ρ defines a Lie algebra homomorphism from $\Gamma(A)$ to the space $\chi^1(P)$ of vector fields on P with the commutator Lie bracket, and

2) for $f \in C^\infty(P)$ and $\omega_1, \omega_2 \in \Gamma(A)$, the following derivation law holds:

$$[\omega_1, f\omega_2] = f[\omega_1, \omega_2] + (\rho(\omega_1)f)\omega_2.$$

The map $\rho : A \to TP$ is called the anchor map of the Lie algebroid A. A *representation* of a Lie algebroid A over P is a vector bundle E over P, together with an $\mathbb{R}$-bilinear map

$$\Gamma(A) \ \times\ \Gamma(E) \longrightarrow \Gamma(E) : \ a \otimes s \longmapsto D_a s.$$

such that for any $a, b \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(P)$,

$$\begin{aligned} (1)\ D_{fa}s &= fD_a s; \\ (2)\ D_a(fs) &= fD_a s + (\rho(a)f)s; \\ (3)\ D_a(D_b s) - D_b(D_a s) &= D_{[a,b]}s. \end{aligned}$$

Example 2.1 Any Lie algebra is a Lie algebroid over a one point space, and its representations are representations of this Lie algebra. The tangent bundle TP is a Lie algebroid over P with the identity map of TP as the anchor map; its representations are vector bundles over P with flat connections. The trivial representation of a Lie algebroid A is, by definition, the representation of A on the trivial line bundle over P with the action

$$D_a f \ = \ \rho(a) \cdot f, \qquad a \in \Gamma(A),\ f \in C^\infty(P).$$

Example 2.2 Recall [Va] that a Poisson structure on a manifold P is a bivector field $\pi \in \chi^2(P)$ such $[\pi, \pi] = 0$, where $[\,,\,]$ is the Schouten bracket on $\chi^\bullet(P)$. A Poisson structure π on P defines a Lie algebroid structure on T^*P over P: the anchor map $\tilde{\pi} : T^*P \to TP$ is

$$\tilde{\pi}(p) : \ T_p^*P \longrightarrow T_pP : \ \alpha_p \longmapsto \alpha_p \,\lrcorner\, \pi(p),$$

and the Lie bracket of 1-forms α and β is given by

$$\begin{aligned} \{\alpha,\beta\} &= d\pi(\alpha,\beta) + \tilde{\pi}(\alpha) \lrcorner d\beta - \tilde{\pi}(\beta) \lrcorner d\alpha & (1)\\ &= -d\pi(\alpha,\beta) + L_{\tilde{\pi}(\alpha)}\beta - L_{\tilde{\pi}(\beta)}\alpha. & (2) \end{aligned}$$

Many geometrical aspects of the Poisson structure π such as its symplectic leaves and transversal Lie algebra structures to the symplectic leaves are reflected in the Lie algebroid structure on T^*P. See [Va].

Let A be a Lie algebroid over P with anchor map ρ. Define $d_A : \Gamma(\wedge^{k-1}A^*) \to \Gamma(\wedge^k A^*)$:

$$\begin{aligned} (d_A\xi)(a_1,...,a_k) &= \sum_i (-1)^{i+1}\rho(a_i)(\xi(a_1,...,\hat{a}_i,...,a_k)) & (3)\\ &+ \sum_{i<j}(-1)^{i+j}\xi(\{a_i,a_j\},...,\hat{a}_i,...,\hat{a}_j,...,a_k). \end{aligned}$$

It is well-defined, and $d_A^2 = 0$. The cohomology of $(\Gamma(\wedge^\bullet A^*), d_A)$ is the *Lie algebroid cohomology* of A (with trivial coefficients), and it is denoted by $H^\bullet(A)$.

When $A = TP$, the Lie algebroid cohomology of A is the de Rham cohomology of P. When P is a point, Lie algebroid cohomology is Lie algebra cohomology. When $A = T^*P$ is the cotangent bundle Lie algebroid of a Poisson manifold (P,π), the Lie algebroid cohomology of A is called the *Poisson cohomology* of (P,π) [Li] [Va], and we will denote it by $H^\bullet(P,\pi)$.

Suppose that E is a representation of A. Let $\Gamma^k(A^*,E) = \Gamma(\wedge^k A^*)\otimes\Gamma(E)$. Regard the representation of A on E as defining a map $D : \Gamma^0(A^*,E) = \Gamma(E) \to \Gamma^1(A^*,E)$ and extend D to a map $D : \Gamma^k(A^*,E) \to \Gamma^{k+1}(A^*,E)$ by the rule

$$D(\xi \otimes s) = d_A\xi \otimes s + (-1)^k \xi \otimes Ds, \quad \xi \in \Gamma(\wedge^k A^*),\ s \in \Gamma(E).$$

Then $D^2 = 0$. The cohomology of $(\Gamma^\bullet(A^*,E),\ D)$ is called the Lie algebroid cohomology of A with coefficients in E, and it is denoted by $H^\bullet(A;E)$.

We will also use the Schouten bracket $[\ ,\ ]$ on $\Gamma(\wedge A)$. It is characterized by the following properties: for $f \in \Gamma(\wedge^0 A) = C^\infty(P),\ a, a_1, a_2 \in \Gamma(A)$ and $X, Y, Z \in \Gamma(\wedge^\bullet A)$,

$$[X,Y] \in \Gamma(\wedge^{|X|+|Y|-1} A); \tag{4}$$

$$[a,\ f] \ = \ \rho(a) \cdot f; \tag{5}$$

$$[a_1,\ a_2] \text{ is the Lie bracket in } \Gamma(A); \tag{6}$$

$$[X,Y] = -(-1)^{(|X|-1)(|Y|-1)}[Y,X]; \tag{7}$$

$$[X, Y \wedge Z] = [X,Y] \wedge Z + (-1)^{(|X|-1)|Y|} Y \wedge [X,Z]. \tag{8}$$

For $a \in \Gamma(A)$, set

$$L_a : \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^k A) : \ L_a X \ = \ [a,\ X],, \ k \geq 0.$$

The same L_a will also denote the operator on $\Gamma(\wedge^\bullet A^*)$ given by

$$(L_a \xi,\ X) \ + \ (\xi,\ L_a X) \ = \ \rho(a) \cdot (\xi,\ X), \qquad \xi \in \Gamma(\wedge A^*),\, X \in \Gamma(\wedge A). \tag{9}$$

Then $L_a = d_{\scriptscriptstyle A} i_a + i_a d_{\scriptscriptstyle A}$. For any $a \in \Gamma(A)$, $X \in \Gamma(\wedge^{top} A)$, and $\xi \in \Gamma(\wedge^{top} A^*)$:

$$L_{fa} X \ = \ f L_a X \ - \ (\rho(a) \cdot f) X \tag{10}$$

$$L_a(fX) \ = \ f L_a X \ + \ (\rho(a) \cdot f) X \tag{11}$$

$$L_{fa} \xi \ = \ f L_a \xi \ + \ (\rho(a) \cdot f) \xi \tag{12}$$

$$L_a(f\xi) \ = \ f L_a \xi \ + \ (\rho(a) \cdot f) \xi. \tag{13}$$

3 The representation of A on Q_A and the modular class of A

Let A be a Lie algebroid over P and let L be a line bundle representation of A. We first assume that L is trivial as a line bundle.

Let s be a nowhere vanishing section of L. For $a \in \Gamma(A)$, define $\theta_s(a) \in C^\infty(P)$ by

$$D_a s \ = \ \theta_s(a) s. \tag{14}$$

It is easy to see that $\theta_s \in \Gamma(A^*)$ is a 1-cocycle with respect to $d_{\scriptscriptstyle A}$. If s_1 is another nowhere vanishing section of L, and if $s_1 = f_1 s$ for a nowhere vanishing function f_1, then

$$\theta_{s_1} \ = \ \theta_s \ + \ d_{\scriptscriptstyle A}(\log |f_1|) \ \in \Gamma(A^*).$$

Thus $[\theta_{s_1}] = [\theta_s] \in H^1(A)$. We denote this class in $H^1(A)$ by θ_L.

If L_1 and L_2 are two line bundle representations of A, we can equip their tensor product $L_1 \otimes L_2$ with the representation of A given by

$$D_a(s_1 \otimes s_2) \;=\; D_a(s_1) \otimes s_2 \;+\; s_1 \otimes D_a(s_2).$$

Then, assuming both L_1 and L_2 both are trivial as line bundles, we have $\theta_{L_1 \otimes L_2} = \theta_{L_1} + \theta_{L_2}$. In particular, $\theta_{L^2} = 2\theta_L$, where $L^2 = L \otimes L$.

For a general L, we define

$$\theta_L \;=\; \frac{1}{2}\theta_{L^2} \in H^1(A).$$

Since the square of any real line bundle over P is trivial as a line bundle, the class θ_{L^2} is defined. By the remark above, θ_L is well-defined.

Definition 3.1 The class $\theta_L \in H^1(A)$ is called the *characteristic class* of A associated to the representation L.

The following proposition follows from our discussions above.

Proposition 3.2 *For any two line bundles L_1 and L_2 with A representations,*

$$\theta_{L_1 \otimes L_2} \;=\; \theta_{L_1} \;+\; \theta_{L_2}.$$

Consider now the line bundle

$$Q_A \;=\; \wedge^{top} A \otimes \; \wedge^{top} T^*P. \tag{15}$$

For $a \in \Gamma(A)$, define $D_a : \Gamma(Q_A) \to \Gamma(Q_A)$ by

$$D_a(X \otimes \mu) \;=\; L_a(X) \otimes \mu \;+\; X \otimes L_{\rho(a)}\mu, \tag{16}$$

where $X \in \Gamma(\wedge^{top} A)$, $\mu \in \Gamma(\wedge^{top} T^*P)$, and $L_{\rho(a)}\mu$ is the Lie derivative of μ in the direction of $\rho(a)$. It follows from (11) and (13) that D_a is well-defined.

Theorem 3.3 *The map*

$$\Gamma(A) \otimes \Gamma(Q_A) \longrightarrow \Gamma(Q_A) : \; a \otimes (X \otimes \mu) \longmapsto D_a(X \otimes \mu)$$

given by (16) defines a representation of A on the line bundle Q_A.

Proof. The proof, using (10) - (13), is straightforward. We omit it.

Q.E.D.

Definition 3.4 We define the *modular class* of A to be the characteristic class associated to the representation of A on the line bundle Q_A, and we denote it by θ_A

Remark 3.5 For readers familiar with the theory of D-modules, we briefly sketch a D-module motivation for the definition of Q_A. Let D_P be the sheaf of differential operators on a manifold P, and recall that $\wedge^{top}T^*P$ is a right module for D_P. The action by a vector field ξ on a form ω is given by $\omega \cdot \xi = -L_\xi(\omega)$ and the action by functions is given by multiplication (see [Bo], VI.3). If M is a right D_P module, it is easy to see that $Hom(\wedge^{top}T^*P, M)$ is a left D_P module under the action of a vector fields ξ on a morphism λ given by $\xi(\lambda)(\omega) = \lambda(\omega \cdot \xi) - (\lambda(\omega)) \cdot \xi$. Recall that D_P is the sheaf of algebras generated by functions and the tangent bundle TP. Replacing TP with a Lie algebroid A, the same construction as above makes $\wedge^{top}A^*$ into a right module for the sheaf of algebras $U(A)$ generated by functions and A. Moreover, the same construction as above shows that if M is a right $U(A)$ module, then $Hom(\wedge^{top}A^*, M)$ is a left $U(A)$ module. However, every Lie algebroid A has an intrinsic right module. Indeed, the anchor map $\rho : A \to TP$ induces a right $U(A)$-module structure on $\wedge^{top}(T^*P)$. Thus, by taking $M = \wedge^{top}(T^*P)$, we get a left $U(A)$ module structure on $Hom(\wedge^{top}A^*, \wedge^{top}T^*P)$. This left $U(A)$ action is equivalent to the representation of A on Q_A.

Remark 3.6 Huebschmann [H] has generalized our construction of Q_A to Lie-Rinehart algebras. For a holomorphic Lie algebroid A, one can define the Lie algebroid cohomology $H^\bullet(A)$ as the hypercohomology of the complex of sheaves:

$$\cdots \to \Omega^{k-1}(A) \to \Omega^k(A) \to \Omega^{k+1}(A) \to \cdots,$$

where $\Omega^k(A) = \Gamma(\wedge^k A^*)$. The existence of the modular class in $H^1(A)$ is then connected to the Chern class of the line bundle Q_A.

Remark 3.7 Another definition of the modular class begins with the observation of Kontsevich [Kt] that a Lie algebroid A can be viewed as a supermanifold A_s, with $\Gamma(\wedge^\bullet A^*)$ as its algebra of functions. The differential d_A defining Lie algebroid cohomology can be considered as a vector field on A_s. On the other hand, a section s_0 of Q_A determines in a natural way a volume element on A_s. The divergence of d_A with respect to this volume element is a function on A_s which is in fact the section θ_{s_0} of A^* representing the modular class.

Example 3.8 When A is an integrable subbundle of TP, the line bundle Q_A is isomorphic to the top exterior power of the conormal bundle to the corresponding foliation $\mathcal{F}$ (assuming the normal bundle is orientable - otherwise twisting by the coorientation bundle is needed), and thus sections of Q_A are transverse measures to $\mathcal{F}$. The representation of A on Q_A is the top exterior power of the Bott connection of the foliation. It is thus appropriate to call the modular class of A the *modular class of the foliation.* (See Appendix A for a generalization of the Bott connection itself.)

Because of Example 3.8, for a general Lie algebroid A, we may think of sections of Q_A as being "*transverse measures to* A".

Example 3.9 When a Lie algebra $\mathfrak{g}$ is considered as a Lie algebroid over a one point space, its modular class is given by the adjoint character $\xi_0 \in \mathfrak{g}^* : (\xi_0, x) = tr(ad_x)$, $x \in \mathfrak{g}$. The modular class of the tangent bundle Lie algebroid is trivial.

Example 3.10 Let $A = P \times \mathfrak{g}$ be the transformation Lie algebroid defined by a Lie algebra homomorphism $\rho : \mathfrak{g} \to \chi^1(P)$. By definition, this is the Lie algebroid structure on the trivial vector bundle $P \times \mathfrak{g}$ on P with the anchor given by ρ and the Lie bracket on $\Gamma(A) \cong C^\infty(P, \mathfrak{g})$ given by

$$\{x, y\} = [x, y] + \rho_x \cdot y - \rho_y \cdot x,$$

where the first term on the right hand side is the pointwise Lie bracket in $\mathfrak{g}$, and $\rho_x \cdot y$ is the Lie derivative of the $\mathfrak{g}$-valued function y in the direction of ρ_x. For this Lie algebroid A, there is a representation of A on each of the line bundles $\wedge^{top} A$ and $\wedge^{top} T^*P$, defined respectively by the adjoint representation of $\mathfrak{g}$ and the action of $\mathfrak{g}$ on $\wedge^{top} T^*P$ by Lie derivatives. The representation of A on Q_A is simply the tensor product of these two. Let μ be a nowhere vanishing density on P. Then the modular class of A is represented by the section of A^* that assigns to each constant section $x \in \mathfrak{g}$ of A the function $(\xi_0, x) - div_\mu(\rho(x))$, where $\xi_0 \in \mathfrak{g}^*$ is the adjoint character of $\mathfrak{g}$, and $div_\mu(\rho(x)) \in C^\infty(P)$ is the divergence of $\rho(x)$ with respect to μ, i.e., $L_{\rho(x)}\mu = div_\mu(\rho(x))\mu$.

4 The case of T^*P when P is a Poisson manifold

Let (P, π) be a Poisson manifold. We first recall the definition of the modular class of π as defined in [We]. Let μ be a nowhere vanishing density on P. The *modular vector field* of π associated to μ is defined in [We] as the vector field θ_μ on P such that

$$L_{\tilde{\pi}(df)}\mu \;=\; \theta_\mu(f)\mu, \qquad \forall f \in C^\infty(P).$$

It is a Poisson vector field (i.e, it is a 1-cocycle for the cotangent bundle Lie algebroid T^*P), and the the corresponding class in the Poisson cohomology space $H^1(P,\pi)$ is defined in [We] as the *modular class* of π. See [B-Z] and [P] for the cases of holomorphic and algebraic Poisson structures respectively.

On the other hand, the cotangent bundle Lie algebroid T^*P defined by π also has a modular class according to our Definition 3.4, and it is also an element in $H^1(P,\pi)$.

Proposition 4.1 *For a Poisson manifold (P,π), the modular class for the cotangent bundle Lie algebroid $A = T^*P$ is twice the modular class of the Poisson structure π as defined in [We].*

Proof. Let μ be a nowhere vanishing density of P, and let θ_μ be the modular vector field of π associated to μ. Find an open cover $\{U_\gamma\}$ of P and nowhere vanishing sections μ_γ of $\wedge^{top}T^*P$ on U_γ such that $\mu^2 = \mu_\gamma^2$ on U_γ. Let $f \in C^\infty(P)$. Then by (16) we have on U_γ:

$$D_{df}\mu^2 \;=\; D_{df}(\mu_\gamma \otimes \mu_\gamma) \;=\; \{df, \mu_\gamma\} \otimes \mu_\gamma \;+\; \mu_\gamma \otimes L_{\tilde{\pi}(df)}\mu_\gamma,$$

where $\{df, \mu_\gamma\}$ is the Schouten bracket between df and μ_γ. It is easy to see from (1) that $\{df, \mu_\gamma\} = L_{\tilde{\pi}(df)}\mu_\gamma$. Thus

$$D_{df}\mu^2 \;=\; 2(L_{\tilde{\pi}(df)}\mu_\gamma)\otimes\mu_\gamma \;=\; L_{\tilde{\pi}(df)}\mu_\gamma^2 \;=\; L_{\tilde{\pi}(df)}\mu^2 \;=\; 2\theta_\mu(f)\mu^2.$$

Hence the modular class of $A = T^*P$ is $2[\theta_\mu] \in H^1(P,\pi)$.

Q.E.D.

The proof of the following proposition is straightforward. We omit it.

Proposition 4.2 *Given a representation of A on the square $M \otimes M = M^2$ of a line bundle M, there is a representation of A on M itself defined as follows: let a be a section of A, and* *let s be a section of M. Let U be an open subset of P over which t is a nowhere vanishing section of M. Write $s = ft$ for some function f defined on U. Define*

$$(D_a s)|_U \;=\; (\rho(a)\cdot f)t \;+\; \frac{1}{2}\frac{D_a(t^2)}{t^2}s. \tag{17}$$

This is well-defined, and it defines a representation of A on M whose square is the original representation of A on M^2.

Proposition 4.2 leads to a representation of T^*P on $\wedge^{top}T^*P$, explicitly given in the next theorem. Formula (20) has also been independently discovered by Ping Xu in [Xu]. See also [P] for the case of algebraic Poisson structures.

Theorem 4.3 *Let P be a Poisson manifold with Poisson bivector field π. For a 1-form α and a top degree form μ on P, define*

$$D_\alpha\mu \;=\; \{\alpha,\ \mu\} \;-\; (\pi,\ d\alpha)\mu, \tag{18}$$

where $\{\alpha,\mu\}$ is the Schouten bracket between α and μ, and $(\pi, d\alpha)$ is the pairing between π and $d\alpha$. Then (18) defines a representation of T^*P on $\wedge^{top}T^*P$. Alternatively,

$$\begin{aligned} D_\alpha\mu \;&=\; L_{\tilde{\pi}(\alpha)}\mu \;+\; (\pi,\ d\alpha)\mu &(19)\\ &=\; \alpha\wedge d i_\pi\mu. &(20)\end{aligned}$$

Proof. It is straightforward to show that $D_{f\alpha}\mu = fD_\alpha\mu$ and $D_\alpha(f\mu) = (\tilde{\pi}(\alpha)\cdot f)\mu + fD_\alpha\mu$ for any $f \in C^\infty(P), \alpha \in \Omega^1(P)$ and $\mu \in \Omega^{top}(P)$. If β is another 1-form on P,

$$\begin{aligned} D_\alpha D_\beta\mu \;-\; D_\beta D_\alpha\mu \;&=\; D_\alpha(\{\beta,\ \mu\} - (\pi,\ d\beta)\mu) \;-\; D_\beta(\{\alpha,\ \mu\} - (\pi, d\alpha)\mu)\\ &=\; \{\alpha,\ \{\beta,\ \mu\}\} \;-\; (\pi,\ d\alpha)\{\beta,\ \mu\}\\ &\quad -\tilde{\pi}(\alpha)\cdot(\pi,\ d\beta)\mu \;-\; (\pi,\ d\beta)\{\alpha,\ \mu\} \;+\; (\pi,\ d\beta)(\pi,\ d\alpha)\mu\\ &\quad -\{\beta,\ \{\alpha,\ \mu\}\} \;+\; (\pi,\ d\beta)\{\alpha,\ \mu\}\\ &\quad +\tilde{\pi}(\beta)\cdot(\pi,\ d\alpha)\mu \;+\; (\pi,\ d\alpha)\{\beta,\ \mu\} \;-\; (\pi,\ d\alpha)(\pi,\ d\beta)\mu\\ &=\; \{\{\alpha,\ \beta\},\ \mu\} \;-\; \tilde{\pi}(\alpha)\cdot(\pi,\ d\beta)\mu \;+\; \tilde{\pi}(\beta)\cdot(\pi,\ d\alpha)\mu\\ &=\; \{\{\alpha,\ \beta\},\ \mu\} \;-\; (\pi,\ d\{\alpha,\ \beta\})\mu\\ &=\; D_{\{\alpha,\ \beta\}}\mu. \end{aligned}$$

In the last step, we used the fact that $\tilde{\pi}(\alpha)\cdot(\pi, d\beta) - \tilde{\pi}(\beta)\cdot(\pi, d\alpha) = (\pi,\ d\{\alpha,\beta\})$. Indeed,

$$\begin{aligned}
\text{l.h.s} &= ([\tilde{\pi}(\alpha),\ \pi],\ d\beta)\ +\ (\pi,\ L_{\tilde{\pi}(\alpha)}d\beta)\ -\ ([\tilde{\pi}(\beta),\ \pi],\ d\alpha)\ -\ (\pi,\ L_{\tilde{\pi}(\beta)}d\alpha) \\
&= -([\pi,\ \tilde{\pi}(\alpha)],\ d\beta)\ +\ ([\pi,\ \tilde{\pi}(\beta)],\ d\alpha)\ +\ (\pi,\ d\{\alpha,\beta\}) \\
&= -(\tilde{\pi}(d\alpha),\ d\beta)\ +\ (\tilde{\pi}(d\beta),\ d\alpha)\ +\ (\pi,\ d\{\alpha,\beta\}) \\
&= (\pi,\ d\{\alpha,\beta\}) \\
&= \text{r.h.s.}
\end{aligned}$$

Here we used Formula (2) and the fact that $\tilde{\pi} : \Omega^{\bullet}(P) \to \chi^{\bullet}(P)$ intertwines the operators d and $d_\pi = [\pi,\ \bullet]$.

To show that $D_\alpha\mu$ is also given by (19), we first observe that the right hand side of (19) is $C^\infty(P)$-linear in α. Since (19) holds when α is exact, it holds for an arbitrary α. To show (20), we start with $\alpha \wedge i_\pi\mu = -i_{\tilde{\pi}(\alpha)}\mu$ and apply d to both sides to get

$$(\pi,\ d\alpha)\mu\ -\ \alpha\wedge di_\pi\mu\ =\ -L_{\tilde{\pi}(\alpha)}\mu.$$

Formula (20) now follows from (19).

Q.E.D.

The characteristic class of the Lie algebroid T^*P associated to this representation on $\wedge^{top}T^*P$ (or the one on the density bundle $|\wedge^{top} T^*P|$ whose existence also comes from Proposition 4.2) is now the same as the modular class of the Poisson structure π defined in [We].

We now consider the Lie algebroid cohomology spaces $H^\bullet(T^*P)$, $H^\bullet(T^*P;\, \wedge^{top}T^*P)$, and $H^\bullet(T^*P;\, (\wedge^{top}T^*P)^2)$. The corresponding cochain complexes will respectively be denoted by (C, δ_π), $(C_1, \delta_\pi^{'})$ and $(C_2, \delta_\pi^{''})$. Of course, $C^k = \chi^k(P)$ for any $0 \le k \le n = \dim P$, and $\delta_\pi(V) = [\pi, V]$ is the Schouten bracket. The proof of the following lemma is straightforward.

Lemma 4.4 *Assume that P is orientable. Let μ_0 be a volume form on P, and let θ_0 be the modular vector field defined by μ. Identify both C_1 and C_2 with $\chi(P)$ by*

$$V \longmapsto V\otimes\mu_0 \in C_1^k, \quad \text{and} \quad V \longmapsto V\otimes\mu_0\otimes\mu_0 \in C_2^k, \quad V \in \chi^k(P).$$

Then the operators $\delta_\pi^{'}$ and $\delta_\pi^{''}$ respectively become

$$\delta_\pi^{'}(V)\ =\ [\pi, V]\ +\ \theta_0\wedge V, \quad \text{and} \quad \delta_\pi^{''}(V)\ =\ [\pi, V]\ +\ 2\theta_0\wedge V.$$

Proposition 4.5 *With the same assumptions as in Lemma 4.4, we have, for any $V \in \chi^k(P)$,*

$$[\pi, V] \lrcorner \mu_0 = (-1)^{k-1}(i_\pi d - di_\pi + i_{\theta_0})(V \lrcorner \mu_0); \tag{21}$$

$$([\pi, V] + \theta_0 \wedge V) \lrcorner \mu_0 = (-1)^{k-1}(i_\pi d - di_\pi)(V \lrcorner \mu_0); \tag{22}$$

$$([\pi, V] + 2\theta_0 \wedge V) \lrcorner \mu_0 = (-1)^{k-1}(i_\pi d - di_\pi - i_{\theta_0})(V \lrcorner \mu_0). \tag{23}$$

Proof. We only need to prove (21). Introduce the operator

$$b_{\mu_0} : \ \chi^k(P) \longrightarrow \chi^{k-1}(P) : \ (b_{\mu_0} V) \lrcorner \mu_0 = (-1)^k d(V \lrcorner \mu_0). \tag{24}$$

Clearly $b_{\mu_0}^2 = 0$. It follows from the definition of the Schouten bracket that

$$b_{\mu_0}(V_1 \wedge V_2) = b_{\mu_0}(V_1) \wedge V_2 + (-1)^{|V_1|} V_1 \wedge b_{\mu_0}(V_2) + (-1)^{|V_1|}[V_1, \ V_2], \quad V_1, V_2 \in \chi^\bullet(P). \tag{25}$$

Moreover, it is clear from (20) that $b_{\mu_0}\pi = \theta_0$. Thus,

$$\begin{aligned} \text{r.h.s. of (21)} &= (b_{\mu_0}(V \wedge \pi) - b_{\mu_0} V \wedge \pi) \lrcorner \mu_0 \\ &= ((-1)^k V \wedge b_{\mu_0}\pi + (-1)^k [V, \pi]) \lrcorner \mu_0 \\ &= (\theta_0 \wedge V + [\pi, V]) \lrcorner \mu_0 \\ &= \text{l.h.s. of (21)}. \end{aligned}$$

Q.E.D.

We call $\partial_\pi = i_\pi d - di_\pi : \Omega^{k+1}(P) \to \Omega^k(P)$ the Koszul-Brylinski operator. It follows from $[\pi, \pi] = 0$ that $\partial_\pi^2 = 0$. The homology of the chain complex $(\Omega^\bullet(P), \partial_\pi)$ is called the *Poisson homology* of (P, π). We will denote it by $H_\bullet(P)$.

Corollary 4.6 *For any Poisson manifold (P, π), The Poisson homology of a Poisson manifold (P, π) is isomorphic to the Lie algebroid cohomology $H^\bullet(T^*P; \wedge^{top} T^*P)$.*

Proof. Let $n = \dim P$. For each $0 \leq k \leq n$, define $\tau : C_1^k = \chi^k(P) \otimes \Omega^n(P) \longrightarrow \Omega^{n-k}(P) : V \otimes \mu \longmapsto V \lrcorner \mu$. We claim that $\tau \delta_\pi'(V \otimes \mu) = (-1)^{k-1} \partial_\pi \tau(V \otimes \mu)$ for any $V \otimes \mu \in C_1^k$. Indeed, since all the operators involved in this formula are local, we only need to prove it locally. By choosing a local nowhere vanishing section of $\wedge^{top} T^*P$, we see that the desired formula comes from (22).

Q.E.D.

For the rest of this section, we consider modular classes for Poisson Lie groups and (some of) their homogeneous spaces. Let (G, π) be a connected Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g},\ \mathfrak{g}^*)$ ([STS] [L-W]). Let $\xi_0 \in \mathfrak{g}$ and $x_0 \in \mathfrak{g}$ be the adjoint characters of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively, i.e., for any $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$,

$$(\xi_0,\ x) \ = \ tr(ad_x), \qquad (x_0,\ \xi) \ = \ tr(ad_\xi).$$

Denote by x_0^l (resp. x_0^r) the left (resp. right) invariant vector fields on G with value x_0 at the identity element e. Denote by ξ_0^r the right invariant 1-form on G whose value at e is ξ_0. Recall that the right dressing vector field [L-W] [STS] defined by ξ_0 is the vector field $\sigma_{\xi_0} = \tilde{\pi}(\xi_0^r)$.

Proposition 4.7 *For a Poisson Lie group (G, π), the modular vector field of π associated to a right invariant volume form μ^r on G is*

$$\theta_{\mu^r} \ = \ \frac{1}{2}(x_0^l \ + \ x_0^r \ - \ \sigma_{\xi_0}). \tag{26}$$

The modular vector field of π associated to a left invariant volume form μ^l is

$$\theta_{\mu^l} \ = \ \frac{1}{2}(x_0^l \ + \ x_0^r \ + \ \sigma_{\xi_0}). \tag{27}$$

Proof. Let α be a right invariant 1-form on G. The Schouten bracket $\{\alpha,\ \mu^r\}$ is also right invariant, and is equal to $(x_0^r, \alpha)\mu^r$. Consider the vector field v on G defined by

$$(v,\ \alpha) \ = \ \frac{1}{2}(x_0^r - x_0^l \ + \ \sigma_{\xi_0},\ \alpha) \ - \ (\pi,\ d\alpha),$$

where α is a right invariant 1-form on G. It remains to show that $v = 0$. Since π is multiplicative, i.e., $\pi(gh) = l_g\pi(h) + r_h\pi(g)$ and since ξ_0 is Ad_G-invariant, we know that v is multiplicative, i.e., $v(gh) = l_g v(h) + r_h v(g)$ for $g, h \in G$. Since $v(e) = 0$, we know that $v = 0$ because the linearization of v at e is zero. The case for μ^l is similarly proved.

Q.E.D.

Remark 4.8 1) The vector fields x_0^l and x_0^r are Poisson vector fields because x_0 is a character of $\mathfrak{g}^*$. Let f_0 be the function on G defined by $f_0(g) := \det(Ad_g \in \mathrm{End}(\wedge^{top}\mathfrak{g}))$. Then $d(\log|f_0|) = \xi_0^r$. Thus $\sigma_{\xi_0} = \tilde{\pi}(d(\log|f_0|))$ is a Hamiltonian vector field.

2) If we identify the cotangent bundle T^*G with the trivial bundle $G\times\mathfrak{g}^*$ by right translations, the Lie algebroid structure on T^*G becomes that of the transformation Lie algebroid defined by the infinitesimal right dressing action of $\mathfrak{g}^*$ on G (see [L-W]). Proposition 4.7 can then also be proved as a corollary of Example 3.10.

Assume now that $H\subset G$ is a closed Poisson Lie subgroup of G [STS], so that $\mathfrak{h}^\perp = \{\xi\in\mathfrak{g}^* : (\xi,x)=0\,\forall x\in\mathfrak{h}\}$ is an ideal in $\mathfrak{g}^*$.

The quotient space G/H has a unique Poisson structure such that the projection $j: G\to G/H : g\mapsto gH$ is a Poisson map. Assume that there exists a G-invariant volume form μ_0 on G/H. We wish to determine the modular vector field θ_{μ_0} on G/H defined by μ_0.

Set $\tilde\mu_0 = j^*\mu_0$. It is a left invariant l-form on G, where $l=\dim(G/H)$. Choose any $\mu_1\in\wedge^{n-l}\mathfrak{g}^*$, where $n=\dim G$, such that $\mu=\mu_1\wedge\tilde\mu_0$ is a left invariant volume form on G. We have shown that the modular vector field θ_μ on G defined by μ is given by (27).

Proposition 4.9 *The modular vector field θ_{μ_0} on G/H defined by μ_0 is given by*

$$\theta_{\mu_0} \;=\; j_*(\theta_\mu) \;=\; \frac{1}{2}j_*(x_0^l \;+\; x_0^r \;+\; \sigma_{\xi_0}). \tag{28}$$

Proof. Let $\gamma_1,...,\gamma_l$ be a basis for $\mathfrak{h}^\perp\subset\mathfrak{g}^*$. Denote by the same letters the corresponding left invariant 1-forms on G. Assume that $\mu_1=\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_t$, where $t=\dim H$ and the ξ_i's are in $\mathfrak{g}^*$.

Let α be an arbitrary 1-form on G/H. Let $\tilde\alpha = j^*\alpha$. Write $\tilde\alpha=\sum_{i=1}^l f_i\gamma_i$, where the f_i's are functions on G. Then

$$\begin{aligned}\{\tilde\alpha,\,\mu\} &= \{\tilde\alpha,\,\mu_1\}\wedge\tilde\mu_0 \;+\; \mu_1\wedge\{\tilde\alpha,\,\tilde\mu_0\}\\ &= \sum_{j=1}^t \xi_1\wedge\cdots\wedge\xi_{j-1}\wedge\left(\sum_{i=1}^l (f_i\{\gamma_i,\,\xi_j\} \;-\; (\tilde\pi(\xi_j)\cdot f_i)\gamma_i)\right)\wedge\xi_{j+1}\wedge\cdots\wedge\xi_t.\end{aligned}$$

Since $\mathfrak{h}^\perp$ is an ideal of $\mathfrak{g}^*$, we have $\{\gamma_i,\,\xi_j\}\in\mathfrak{h}^\perp$ for all i,j. Thus $\{\tilde\alpha,\mu_1\}\wedge\tilde\mu_0=0$, and

$$\{\tilde\alpha,\,\mu\} \;=\; \mu_1\wedge\{\tilde\alpha,\,\tilde\mu_0\} \;=\; \mu_1\wedge j^*\{\alpha,\,\mu_0\}.$$

Using π_G and π to denote respectively the Poisson bi-vector fields on G and G/H. Then,

$$\begin{aligned}\theta_\mu(\tilde\alpha)\mu &= \{\tilde\alpha,\,\mu_1\wedge\tilde\mu_0\} \;-\; (\pi_G,\,d\tilde\alpha)\mu_1\wedge\tilde\mu_0\\ &= \mu_1\wedge j^*\{\alpha,\,\mu_0\} \;-\; j^*(\pi,\,d\alpha)\mu_1\wedge\tilde\mu_0\\ &= \mu_1\wedge j^*(\theta_{\mu_0}(\alpha)\mu_0)\\ &= j^*(\theta_{\mu_0}(\alpha))\mu.\end{aligned}$$

Thus $\theta_\mu(\tilde{\alpha}) = j^*(\theta_{\mu_0}(\alpha))$. This shows that $j_*\theta_\mu$ is well-defined and that

$$\theta_{\mu_0} \;=\; j_*\theta_\mu \;=\; \frac{1}{2}j_*(x_0^l \;+\; x_0^r \;+\; \sigma_{\xi_0}).$$

Q.E.D.

Example 4.10 Let K be a compact semi-simple Lie group and let T be a maximal torus of K. There is (after fixing a choice of positive roots) a naturally defined Poisson structure on K making it into a Poisson Lie group [L-W]. T is a Poisson Lie subgroup of K, and the resulting Poisson structure on K/T is called the Bruhat-Poisson structure [L-W]. it follows from Proposition 4.9 that the modular vector field of the Bruhat-Poisson structure defined by a K-invariant volume form on K/T is the vector field defined by iH_ρ in the Lie algebra $\mathfrak{t}$ of T, where ρ is half of the sum of all the positive roots, and iH_ρ denotes the element in $\mathfrak{t}$ corresponding to $i\rho$ under the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$ via the Killing form. This result plays a central role in [E-L], where we give Poisson geometrical interpretations of the Kostant harmonic forms on K/T.

5 A cohomology pairing

Let A be a Lie algebroid of rank r. In this section, we show that there is a natural pairing between $H^k(A)$, the k-th Lie algebroid cohomology of A with trivial coefficients, and $H^{r-k}(A, Q_A)$, the $(r-k)$-th Lie algebroid cohomology of A with coefficients in $Q_A = \wedge^r A \otimes \wedge^{top} T^*P$ for each $0 \leq k \leq r$. We will assume that P is compact and orientable. For a general P, we need to consider the "compactly supported" Lie algebroid cohomology of A and replace $\wedge^{top} T^*P$ by $|\wedge^{top} T^*P|$, the density bundle of P. A similar pairing for Lie-Rinehart algebras is studied in [H].

Set $C = \oplus_{k=0}^r C^k = \oplus_{k=0}^r \Gamma(\wedge^k A^*)$ and $\tilde{C} = \oplus_{k=0}^r \tilde{C}^k = \oplus_{k=0}^l C^k \otimes \Gamma(Q_A)$. Let $\tilde{d}_A : \tilde{C}^k \to \tilde{C}^{k+1}$ be the coboundary operator such that the cohomology of $(\tilde{C}, \tilde{d}_A)$ is $H^\bullet(A, Q_A)$. Fix an orientation of P. Then integrating with respect to this orientation gives rise to the non-degenerate pairing between C^k and $\tilde{C}^{r-k}$:

$$((\xi,\ \eta \otimes X \otimes \mu)) \stackrel{def}{=} \int_P (\xi \wedge \eta,\ X)\mu. \tag{29}$$

Theorem 5.1 *Assume that P is compact orientable. For $\xi \in C^{k-1}$ and $\eta \otimes X \otimes \mu \in \tilde{C}^{r-k}$,*

$$((d_A\xi,\ \eta \otimes X \otimes \mu)) \;+\; (-1)^{k-1}((\xi,\ \tilde{d}_A(\eta \otimes X \otimes \mu))) \;=\; 0. \tag{30}$$

Consequently, there is an induced pairing between $H^k(A)$ and $H^{r-k}(A, Q_A)$.

Proof. Let $c = (\xi \wedge \eta)\otimes X\otimes\mu \in \tilde{C}^{r-1}$. Then

$$\tilde{d}_A(c) \;=\; d_A\xi \wedge (\eta\otimes X\otimes\mu) \;+\; (-1)^{|\xi|}\xi \wedge \tilde{d}_A(\eta\otimes X\otimes\mu) \in \tilde{C}^r.$$

Theorem 5.1 now follows from the following Stokes' Theorem.

Q.E.D.

Stokes' Theorem. *Identify* $\tilde{C}^r = \Gamma(\wedge^r A^*\otimes \wedge^r A\otimes \wedge^n T^*P)$ *with* $\Omega^n(P)$, *where* $n = \dim P$, *by pairing the factors in* $\wedge^r A^*$ *and* $\wedge^r A$ *pointwise. We have, for any* $c = \xi\otimes X\otimes\mu \in \tilde{C}^{r-1}$,

$$\tilde{d}_A(c) \;=\; (-1)^{r-1} d(\rho(\xi \,\lrcorner\, X) \,\lrcorner\, \mu). \tag{31}$$

Consequently, $\int_P \tilde{d}_A(c) = 0$.

Proof. We only need to prove (31) locally. Let X_0 and μ_0 be nowhere vanishing sections of $\wedge^r A$ and $\wedge^n T^*P$ respectively. Set $s_0 = X_0\otimes\mu_0$. Let θ_0 be the section of A^* over U such that $\theta_0(a)s_0 = D_a s_0$ for any $a \in \Gamma(A)$. Write $X\otimes\mu = fX_0\otimes\mu_0 = fs_0$ over U. Then,

$$\tilde{d}_A(c) \;=\; (d_A(f\xi) + \theta_0 \wedge f\xi,\; X_0)\mu_0.$$

Set $a = f\xi \,\lrcorner\, X_0$. It follows from the definition of θ_0 and Formula (16) that

$$(\theta_0 \wedge f\xi,\; X_0)\mu_0 \;=\; -(d_A(f\xi),\; X_0)\mu_0 \;+\; (-1)^{r-1}L_{\rho(a)}\mu_0.$$

Hence,

$$\tilde{d}_A(c) \;=\; (-1)^{r-1}L_{\rho(a)}\mu \;=\; (-1)^{r-1}d(\rho(\xi \,\lrcorner\, X) \,\lrcorner\, \mu).$$

This proves (31). It now follows from Stokes' Theorem for de Rham cohomology that

$$\int_P \tilde{d}_A(c) \;=\; 0.$$

Q.E.D.

Example 5.2 When A is a Lie algebra $\mathfrak{g}$ considered as a Lie algebroid over a one point space, the pairing $(\!(\;,\;)\!)$ in Theorem 5.1 is non-degenerate and it gives rise to Poincaré duality for Lie algebra cohomology.

When $A = TP$ is the tangent bundle Lie algebroid for a compact orientable manifold P, the pairing $(\!(\;,\;)\!)$ in Theorem 5.1 is the one obtained by integrating the wedge product of k-forms and $(n-k)$-forms. This, of course, is non-degenerate and gives the well-known Poincaré duality for de Rham cohomology of P.

Example 5.3 For a general Lie algebroid A of rank r, the pairing $(\!(\, , \,)\!)$ between $H^k(A)$ and $H^{r-k}(A, Q_A)$ is not necessarily non-degenerate. For example, the vector field $v_0 = x^N \frac{d}{dx}$, where $N \geq 2$ is an integer, defines an action of the 1-dimensional Lie algebra $\mathfrak{g} = \mathbb{R}^1$ on $P = \mathbb{R}$. Let $A = \mathbb{R}^1 \times \mathbb{R}^1$ be the transformation Lie algebroid. Then $H^1(A)$ is 1-dimensional, but $H^1_c(A; Q_A) \cong \Omega^1_c(\mathbb{R}^1)/(\mathfrak{g} \cdot \Omega^1_c(\mathbb{R}^1))$ is at least N-dimensional. The subscript c here stands for "compactly supported".

Corollary 5.4 *Let (P, π) be an n-dimensional compact orientable Poisson manifold. Then, for each $0 \leq k \leq n$, there a pairing between the Poisson cohomology $H^k(P, \pi)$ and the Lie algebroid cohomology $H^{n-k}(T^*P; (\wedge^{top} T^*P)^2)$ coming from the the pairing*

$$(\!(U, \, V \otimes \eta \otimes \mu)\!) \;=\; \int_P (U \wedge V, \, \eta) \mu$$

between $\chi(P)$ and $\chi(P) \otimes \Omega^n(P) \otimes \Omega^n(P)$.

Corollary 5.5 *Let P be an n-dimensional orientable Poisson manifold. For a differential form α and a compactly supported differential form β, define*

$$(\alpha, \; \beta) \;=\; \int_P \alpha \wedge \beta.$$

Let $\partial_\pi = i_\pi d - d i_\pi : \Omega^k(P) \to \Omega^{k-1}(P)$ be the Koszul-Brylinski operator. Then, for any forms α and β with $|\alpha| + |\beta| - 1 = n$,

$$(\partial_\pi \alpha, \; \beta) \; + \; (-1)^{(|\alpha|-1)} (\alpha, \; \partial_\pi \beta) \;=\; 0. \tag{32}$$

Thus we get an induced pairing between the Poisson homology spaces $H_k(P)$ and $H_{n-k}(P)$.

Proof. Fix a volume form μ_0 of P and let θ_0 be the modular vector field defined by μ_0. Let $U \in \chi^{n-|\alpha|}(P)$ and $V \in \chi^{n-|\beta|}(P)$ be such that $\alpha = U \lrcorner \mu_0$ and $\beta = V \lrcorner \mu_0$. Let $(\!(\quad)\!)$ be the pairing on $\chi(P)$ given by

$$(\!(V_1, \; V_2)\!) \;=\; \int_P (V_1 \wedge V_2, \; \mu_0) \mu_0.$$

Then we know from Corollary 5.4 that

$$(\!([\pi, \; U], \; V)\!) \; + \; (-1)^{|U|} (\!(U, \; [\pi, V] + 2\theta_0 \wedge V)\!) \;=\; 0.$$

It follows that

$$\begin{aligned} & (\!([\pi, \; U] + \theta_0 \wedge U, \; V)\!) \; + \; (-1)^{|U|} (\!(U, \; [\pi, V] + \theta_0 \wedge V)\!) \\ = \;\; & (\!([\pi, \; U], \; V)\!) \; + \; (-1)^{|U|} (\!(U, \; [\pi, V] + 2\theta_0 \wedge V)\!) \;=\; 0. \end{aligned}$$

But we know from (22) that

$$
\begin{aligned}
(\partial_\pi \alpha,\ \beta) &= (-1)^{|U|-1}(([\pi,\ U] + \theta_0 \wedge U,\ V)), \\
(\alpha,\ \partial_\pi \beta) &= (-1)^{|V|-1}((U,\ [\pi,\ V] + \theta_0 \wedge V)).
\end{aligned}
$$

This shows (32).

Q.E.D.

6 Appendix A: The adjoint "representation" of a Lie algebroid

In this appendix, we will describe a construction which includes as special cases the adjoint representation of a Lie algebra and the (dual of the) flat "Bott" connection on the normal bundle to a foliation. Given a Lie algebroid A, we will construct a "representation up to homotopy" of A on the "formal difference" $A \ominus TP$, taken roughly in the sense of K-theory. The highest exterior power of this object is the representation of A on Q_A described in Section 3.

The idea here is like that in the construction of the representation of A on Q_A, namely that neither the Lie derivative action of A on itself nor on TP is a representation, but that "the anomalies cancel." The following discussion makes this idea precise.

Definition 6.1 Let A be a Lie algebroid over P and let (E, ∂) be a bundle of $\mathbb{Z}_2$-graded complexes over P; i.e., E is a $\mathbb{Z}_2$-graded vector bundle over P and ∂ is a bundle map of degree 1 with $\partial^2 = 0$. A *representation up to homotopy* of A on (E, ∂) is an $\mathbb{R}$-bilinear map

$$
\Gamma(A)\ \times\ \Gamma(E) \longrightarrow \Gamma(E):\ a \otimes s \longmapsto D_a s,
$$

such that the operators D_a preserve the grading and commute with the action of ∂ on sections, and such that the properties

$$
\begin{aligned}
(2)\ D_a(fs) &= fD_a s + (\rho(a)f)s; \\
(3)\ D_a(D_b s) - D_b(D_a s) &= D_{[a,b]} s.
\end{aligned}
$$

of a representation hold, while property

$$
(1)\ D_{fa} s = f D_a s
$$

holds only up to homotopy, in the sense that for each $a \in \Gamma(A)$ and $f \in C^\infty(P)$ there is a bundle map $I(a,f) : E \longrightarrow E$ of degree 1 such that

$$(1') \;\; D_{fa}s \; = \; fD_a s + I(a,f)\partial s + \partial I(a,f)s.$$

For a bundle (E, ∂) of complexes over P, the homologit$H(E)$ is not a vector bundle unless ∂ has constant rank, but the sections of $H(E)$ still form a $C^\infty(P)$-module, so the notion of a representation of A on $H(E)$ still makes sense. The *determinant line bundle* $\wedge^{top}E$ of the graded vector bundle $E = E_0 \oplus E_1$ is defined to be $\wedge^{top}E_0^* \otimes \wedge^{top}E_1$. A standard argument shows that the fibres of $\wedge^{top}E$ are isomorphic to those of $\wedge^{top}H(E)$, so that the latter determinant bundle inherits a smooth structure from the former.

Proposition 6.2 *A representation up to homotopy of a Lie algebroid A on a bundle (E, ∂) of $\mathbb{Z}_2$-graded complexes induces representations (not just up to homotopy) on the homology $H(E)$ and the determinant line bundle $\wedge^{top}E$.*

Proof. Since each D_a commutes with ∂, it induces an operator $H(D_a)$ on homology. Properties (2) and (3) of the D_a's are inherited by the induced operators. To verify (1), we let s be a cycle and find from $(1')$ that $D_{fa}s - fD_a s$ is a boundary.

When ∂ has constant rank $H(E)$ is a smooth bundle, and we can construct a representation on $\wedge^{top}E$ by using $\wedge^{top}H(E)$. For the general case, we work with E itself.

First of all, we note that a family of operators D_a on sections of a vector bundle V satisfying (2) and (3) can be extended in a unique way to a family of operators on all the tensors over V, satisfying the same identities, by requiring that the operators be derivations with respect to tensor product and commute with contractions. This is done just as is usually done for Lie derivatives or covariant derivatives. In particular, we obtain operators, also denoted by D_a, on E^*, $\wedge^{top}E_0^*$, $\wedge^{top}E_1$, and $\wedge^{top}E_0^* \otimes \wedge^{top}E_1$.

For instance, the operators on E^* are defined by

$$(D_a\omega, s) = \rho(a) \cdot (\omega, s) - (\omega, D_a s),$$

from which it follows that we have a representation up to homotopy on the dual complex (E^*, ∂^*), using the bundle maps $-I(a,f)^*$ to satisfy $(1')$.

To compute the operators on $\wedge^{top}E_1$, we use a local basis $t_1, \ldots, t_l$ for the sections of E_1. By the derivation property,

$$D_a(t_1 \wedge \ldots \wedge t_l) = \sum_{r=1}^{l} (t_1 \wedge \ldots \wedge D_a t_r \wedge \ldots \wedge t_l).$$

Then

$$\begin{aligned} D_{fa}(t_1 \wedge \ldots \wedge t_l) &= \sum_{r=1}^{l}(t_1 \wedge \ldots \wedge D_{fa}t_r \wedge \ldots \wedge t_l) \\ &= fD_a(t_1 \wedge \ldots \wedge t_l) \\ &+ \sum_{r=1}^{l}(t_1 \wedge \ldots \wedge (I^{10}(a,f)\partial^{01} + \partial^{10}I^{01}(a,f))t_r \wedge \ldots \wedge t_l), \end{aligned}$$

where the superscript ^{ij} on I or ∂ refers to the part of that bundle map which goes from E_j to E_i.

Expressing $(I^{10}(a,f)\partial^{01} + \partial^{10}I^{01}(a,f))t_r$ in terms of our basis, we see that all but its r'th component is annihilated by another factor in the wedge product, so that we get a simple expression in terms of a trace:

$$D_{fa}(t_1 \wedge \ldots \wedge t_l) = fD_a(t_1 \wedge \ldots \wedge t_l) + \mathrm{Tr}(I^{10}(a,f)\partial^{01} + \partial^{10}I^{01}(a,f))t_1 \wedge \ldots \wedge t_l. \tag{33}$$

Similarly, in terms of a basis $s_1^*, \ldots, s_k^*$ of local sections of E_0^*, we have:

$$D_{fa}(s_1^* \wedge \ldots \wedge s_k^*) = fD_a(s_1^* \wedge \ldots s_k^*) - \mathrm{Tr}(I^{10*}(a,f)\partial^{01*} + \partial^{10*}I^{01*}(a,f))s_1^* \wedge \ldots s_k^*. \tag{34}$$

Combining the two previous equations and using the derivation property with respect to tensor product, we obtain an expression for the action of the D_a's on the determinant bundle:

$$D_{fa}(s_1^* \wedge \ldots s_k^* \otimes t_1 \wedge \ldots \wedge t_l) = fD_a(s_1^* \wedge \ldots s_k^* \otimes t_1 \wedge \ldots \wedge t_l) + K(a,f)s_1^* \wedge \ldots s_k^* \otimes t_1 \wedge \ldots \wedge t_l,$$

where

$$K(a,f) = \mathrm{Tr}(I^{10}(a,f)\partial^{01} + \partial^{10}I^{01}(a,f)) - \mathrm{Tr}(I^{10*}(a,f)\partial^{01*} + \partial^{10*}I^{01*}(a,f)).$$

Using the invariance of the trace under dualization and exchange of factors, we can cancel all the terms in this expression to get $K(a,f) = 0$, so that the operation of A on the determinant bundle satisfies $D_{fa} = fD_a$, and we have a representation.

Q.E.D.

Our principal example of representation up to homotopy is the following. Given the Lie algebroid A with anchor $\rho : A \to TP$, we let $E_0 = TP$, $E_1 = A$, $\partial^{01} = \rho$, and $\partial^{10} = 0$. We call E the *normal complex* of the Lie algebroid and denote it by $N(A)$. The determinant bundle of the normal complex is what we have called Q_A.

We define the operators D_a on $\Gamma(E)$ for $a \in \Gamma(A)$ by $D_a b = [a, b]$ for $b \in \Gamma(A)$ and by $D_a u = [\rho(a), u]$ for $u \in \Gamma(TP)$, where the last bracket is the usual bracket of vector fields.

Proposition 6.3 *The operators defined above form a representation up to homotopy of the Lie algebroid A on $N(A)$.*

Proof. The D_a are chain maps because ρ defines a Lie algebroid homomorphism. Properties (2) and (3) in the definition of a representation are standard facts in the differential calculus on Lie algebroids (see Section 2). For property ($1'$), we note as in Section 3 that for the action on A we have

$$D_{fa} b = f D_a b - (\rho(b) \cdot f) a$$

and for the action on TP we have

$$D_{fa} u = f D_a u - (u \cdot f) \rho(a).$$

If we define the homotopy operators by $I^{10}(a, f)(u) = -(df(u))a$ and $I^{01}(a, f) = 0$, we derive ($1'$) immediately from the two displayed equations above.

Q.E.D.

When A is a Lie algebra, its operation on $N(A)$ is the adjoint representation, so we call this operation the adjoint "representation" in the general case. When ρ is the inclusion of a subbundle of TP, the homology of the normal complex is just the normal bundle to the corresponding foliation, and we recover the usual flat connection along the leaves. Finally, the associated representation of A on Q_A may now be seen as the "top exterior power of the adjoint representation" for a general Lie algebroid.

7 Appendix B: The adjoint "representation" and modular class of a Lie groupoid

Let G be a Lie groupoid (i.e. a differentiable groupoid) over P. We will denote its target and source maps by α and β, so that the product gh is defined whenever $\beta(g) = \alpha(h)$. The groupoid analog of the Lie algebroid "representation" constructed in the previous section ought to be a representation up to homotopy of G on the normal complex of its Lie algebroid A. There are many possible definitions of the notion of representation up to homotopy for a groupoid, and we have not yet found an optimal one, so we will limit ourselves here to a discussion of the particular case of the adjoint "representation."

Recall that a representation of G on a vector bundle $\lambda : B \to P$ consists of a mapping $(g,r) \mapsto gr$ from $G \times_P B = \{(g,r) \in G \times B | \beta(g) = \lambda(r)\}$ to B which is linear on fibres of B and which satisfies the axioms:

(1) $\lambda(gr) = \alpha(g)$;

(2) $(gh)r = g(hr)$;

(3) $er = r$ when e is an identity element of G.

In the case of Lie algebroids, the distinction between a representation up to homotopy and representation is that $(D_a s)(x)$ can depend on the entire section a in the former case, while it depends only on $a(x)$ in the latter. In fact, for the adjoint "representation", $D_a s(x)$ depends only on the 1-jet of a at x.

Let J^1G be the 1-jet prolongation groupoid of the Lie groupoid G over P; i.e. the elements of J^1G are the 1-jets of submanifolds of G which project diffeomorphically to P under the source and target maps.

Unlike G itself, J^1G has a natural representation on the normal complex $N(A)$ of the Lie algebroid of G; i.e., J^1G has representations on A and TP for which the anchor ρ is an equivariant map. One way to see this is to consider elements of J^1G as special subsets of the tangent bundle groupoid TG over TP. TG acts on itself by left translations, leaving invariant the part T_PG of TG lying over the identity section of G. Identifying TP with a subbundle of T_PG and A with the normal bundle T_PG/TP gives the required representation of J^1G.

For example, if G is the pair groupoid $P \times P$, an element (x,y) of G does not naturally transport tangent vectors from y to x, but an element of J^1G is the graph of an isomorphism from T_yP to T_xP.

We would like to make the action of J^1G descend to G via the natural "forgetful" projection $j : J^1G \to G$. As the example above shows, this is not possible, but it turns out to be possible "up to homotopy" in the sense that, if g_1' and g_2' are two elements of J^1G lying over the same $g \in G$, then the mappings induced by the g_i' between the complexes

$N_{\alpha(g)}A$ and $N_{\beta(g)}A$ are homotopic. It suffices to prove this when g is an identity element at $x \in P$, in which case the "difference" between g_1' and g_2' can be considered as a linear map from T_xP to the fibre A_x of the Lie algebroid, and this map (together with the zero map in the other direction) provides the required homotopy operator.

As a consequence of this homotopy property, the representations of J^1G on A and TP descend to representations of G on the (generally singular) normal bundle $TP/\rho(A)$ to the orbits and isotropy bundle ker ρ, and on the determinant line bundle Q_A. In particular, the latter representation defines an element of the first cohomology of G with values in the multiplicative real numbers. This is the modular class of the groupoid.

In the situation of Example 3.8, where A is an integrable subbundle of TP, we may take G to be the holonomy groupoid of the foliation, in which case we recover the linearized holonomy representation of the foliation and the modular class of the foliation in groupoid cohomology discussed in [Ya] and Chapter IV of [M-S].

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