FOURIER TRANSFORM AND THE IWAHORI-MATSUMOTO INVOLUTION

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0. Introduction

The local Langlands program conjectures a classification of smooth, admissible representations of reductive p–adic groups by orbits of groups on certain geometric parameter spaces. This classification has been realized for unramified representations through work of Kazhdan and Lusztig ([KL], see also [Gi]). This paper computes the action of the Iwahori-Matsumoto involution on unramified representations in terms of Fourier transform on the associated parameter space. The Iwahori-Matsumoto involution plays a role for p–adic groups analogous to the role played by the sign representation in the representation theory of Weyl groups. The relation between Fourier transform and the Iwahori-Matsumoto involution was suggested by Lusztig. For $GL(n)$, this result is a special case of a theorem of Moeglin and Waldspurger concerning the Zelevinsky involution.

The category of unramified representations of a split reductive group is equivalent to the category of representations of the Hecke algebra associated to the Iwahori subgroup, which is a specialization of a certain affine Hecke algebra. We will work with the affine Hecke algebra and its graded version in this paper. Let G be the reductive complex algebraic group which is the Langlands dual of the split reductive group considered above. Let $\mathcal{H}$ be the associated affine Hecke algebra, an algebra over Laurent polynomials $\mathbb{C}[v, v^{-1}]$, and let $\mathfrak{g}$ be the Lie algebra of G. Irreducible representations of $\mathcal{H}$ in which v does not act by a root of unity are indexed by quadruples (s, v_0, x, ρ) up to G-conjugacy, where s is a semisimple element of G, $v_0 \in \mathbb{C}^*$ is not a root of unity, x is nilpotent in $\mathfrak{g}$ with $Ad(s)x = {v_0}^2 \cdot x$, and ρ is an irreducible representation of the component group of the centralizer of x appearing in the Springer module of x [KL] (see [Gr] for the root of unity case). We call the pair (s, v_0) an infinitesimal character of the corresponding representation. The centralizer $Z_G(s)$ acts with finitely many orbits on the vector space

$$\mathfrak{g}(s, v_0) = \{x \in \mathfrak{g} | Ad(s)x = {v_0}^2 \cdot x\}$$

and there is a bijection between irreducible representations of $\mathcal{H}$ with infinitesimal character (s, v_0) and $Z_G(s)$-equivariant irreducible perverse sheaves on $\mathfrak{g}(s, v_0)$ appearing in the Springer sheaf. Fourier transform induces a bijection μ from $Z_G(s)$-equivariant irreducible perverse sheaves on $\mathfrak{g}(s, v_0)$ to the corresponding objects on $\mathfrak{g}(s, v_0)^* \cong \mathfrak{g}(s, {v_0}^{-1})$. On the other hand, the Iwahori-Matsumoto involution of

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$\mathcal{H}$ induces a bijection IM from irreducible representations of $\mathcal{H}$ with infinitesimal character (s, v_0) and those with infinitesimal character (s^{-1}, v_0). In addition, there is a very simple involution κ on $\mathcal{H}$ which is the identity on the level of perverse sheaves, but changes infinitesimal character (s, v_0) to $(s^{-1}, {v_0}^{-1})$.

0.1 Theorem. *$\kappa \circ IM$ and Fourier transform induce the same involution on irreducible representations.*

For technical reasons and to achieve greater generality, we work with the graded affine Hecke algebra $H^*_{\mathcal{L}}$ introduced by Lusztig [Lu1] associated to a cuspidal local system $\mathcal{L}$ on a Levi factor. In section two, following an idea of Ginzburg, we reformulate Lusztig's construction of $H^*_{\mathcal{L}}$ as an Ext-algebra of a certain induced sheaf $K_{\mathcal{L}}$, where Ext is calculated in the equivariant derived category. Since Fourier tranform essentially fixes $K_{\mathcal{L}}$ it induces an involution of $H^*_{\mathcal{L}}$ which we identify with $\kappa \circ IM$ in section three. Our result then follows from the compatibility of Fourier transform with localization of equivariant cohomology, which corresponds to passing from $H^*_{\mathcal{L}}$ to the quotient corresponding to the infinitesimal character (s, v_0). This is proven in section four. In sections five and six, we use the equivalences between irreducible representations of graded and ungraded Hecke algebras established by Lusztig to lift our results to the affine Hecke algebra. In section seven, we confirm that our result is equivalent to the theorem of Moeglin and Waldspurger for the unramified case in $GL(n)$, and discuss other groups. We remark that the argument in section four can be simplified in the case when the Levi factor is a Cartan subalgebra. In this setting, the graded Hecke algebra is the graded version of the affine Hecke algebra.

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1. Equivariant Homology

In this section we review equivariant derived categories and equivariant homology with local system coefficients.

1.1. Equivariant Derived Categories. Let $D^b(X)$ be the derived category of bound-
ed complexes of $\mathbb{C}$–sheaves on an algebraic variety X and let $D^b_c(X)$ denote the full subcategory consisting of complexes whose cohomology groups are constructible sheaves. For an action $a : A \times X \longrightarrow X$ by a linear Lie group A, Bernstein and Lunts ([BL], see also [MV]) have defined a triangulated category $D_A(X)$ with a t–structure ([BBD]) and a forgetful functor $\mathcal{F}^A : D_A(X) \longrightarrow D^b_c(X)$ satisfying the following properties:

a) $\mathcal{F}^A$ preserves t–structures and identifies the heart of $D_A(X)$ with A–equivariant constructible sheaves on X,
b) For an A–map $\pi : X \longrightarrow Y$ the functors $\pi_*, \pi_!, \pi^*, \pi^!$ lift to functors between $D_A(X)$ and $D_A(Y)$. The same is true for the duality functor $\mathbf{D}_X$ and the standard properties (as in [KS] for example) hold for these new functors.
c) If a normal subgroup B of A acts freely on X then $D_A(X) \simeq D_{A/B}(B \setminus X)$.

d) For any closed subgroup B of A there is a canonical "forgetful" functor $\mathcal{F}^A_B : D_A(X) \longrightarrow D_B(X)$.

1.2. Construction of $D_A(X)$. Let $A \subseteq GL(n,\mathbb{C})$ be a closed subgroup and let V_k be the Stiefel variety of n–tuples of independent vectors in $\mathbb{C}^{n+k}$. Since A acts freely on V_k one has a diagram $X \xleftarrow{p_k} X \times V_k \xrightarrow{q_k} A \setminus (X \times V_k)$. An object $\mathcal{S}$ in $D_A(X)$ consists of $\mathcal{S}_0 (= \mathcal{F}^A \mathcal{S}) \in D^b_c(X)$ and a family of $\mathcal{S}_k \in D^b_c(A \setminus (X \times V_k))$, $k \geq 1$, together with isomorphisms $p_k^*(\mathcal{S}_0) \simeq q_k^*(\mathcal{S}_k)$, $k \geq 1$, satisfying certain compatibility conditions corresponding to inclusions $V_k \subseteq V_{k+1}$.

The construction of $D_A(X)$ provides a mechanism for extending theorems on the derived category $D^b_c(X)$ to the equivariant setting since calculations with $\mathcal{S} \in D_A(X)$ are simply calculations with sheaves $\mathcal{S}_k \in D^b_c(A \setminus (X \times V_k))$. This principle is easy to apply (and tacitly assumed) throughout the paper.

1.3. Let $\pi : X \to pt$. The equivariant cohomology and compactly supported equivariant cohomology of a sheaf $\mathcal{S} \in D_A(X)$ are defined as

$$H^*_A(X,\mathcal{S}) = Ext^*_{D_A(pt)}(\mathbb{C}_{pt}, \pi_*\mathcal{S}), \quad H^*_{c,A}(X,\mathcal{S}) = Ext^*_{D_A(pt)}(\mathbb{C}_{pt}, \pi_!\mathcal{S}).$$

If $\mathcal{L}$ is an A–equivariant local system on X and each conected component X_i of X is equidimensional one defines the equivariant Borel–Moore homology by

$$H^A_p(X,\mathcal{L}) = \bigoplus_i H_A^{p-2dim X_i}(X, \mathcal{L} \otimes \omega_{X_i}).$$

Here $\omega_Y = (Y \to pt)^! \mathbb{C}_{pt}$ is the dualizing sheaf on Y. This is the same as the definition in [Lu1].

2. The Hecke algebra and the graded Hecke algebra

We will restate Lusztig's construction of $H^*_{\mathcal{L}}$ from [Lu1] in terms of the equivariant derived category by using an idea of Ginzburg from [Gi].

2.1. Let G be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$ and nilpotent cone $\mathcal{N}$ in $\mathfrak{g}$. Let $P = N \rtimes L$ be the Levi decomposition of a parabolic subgroup and let $\mathfrak{p}, \mathfrak{n}, \mathfrak{l}$ be the corresponding Lie algebras. Let $\mathcal{L}_0$ be a cuspidal L–equivariant local system on a nilpotent orbit $\mathcal{C} \subset \mathfrak{l}$ and let $j : \mathcal{C} \hookrightarrow \mathfrak{l}$ be the inclusion. Since $\mathcal{L}_0$ is cuspidal, it follows that its intersection cohomology extension to $\mathfrak{l}$ is clean [Lu6], i.e., the natural map $j_!(\mathcal{L}_0) \to j_*(\mathcal{L}_0)$ is an isomorphism, and both therefore coincide with the intersection cohomology extension after a shift. We write $\mathcal{L} = j_{!*}(\mathcal{L}_0)$ for the shifted intersection cohomology sheaf. Clearly, $\mathcal{L}$ is also equivariant for the action of $L \times \mathbb{C}^*$ on $\mathfrak{l}$ by $(g,\lambda)y = \lambda^{-2} \cdot {}^g y$.

2.2 The induced sheaf $K_{\mathcal{L}}$. Let $K_{\mathcal{L}}$ be the induced sheaf $Ind^G_L(\mathcal{L})$ on $\mathfrak{g}$, which is defined as follows. Consider the $(P \times \mathbb{C}^*)$–equivariant sheaf $\mathcal{L} \boxtimes \mathbb{C}_{\mathfrak{n}}$ on $\mathfrak{p}$, and let $\dot{\mathcal{L}}$ be the corresponding $(G \times \mathbb{C}^*)$–equivariant sheaf on $\dot{\mathfrak{p}} \stackrel{\text{def}}{=} G \underset{P}{\times} \mathfrak{p}$. There is a projection $\pi : \dot{\mathfrak{p}} \to \mathfrak{g}$ given by $\pi(g,x) = {}^g x$ and $K_{\mathcal{L}} \stackrel{\text{def}}{=} \pi_* \dot{\mathcal{L}} \in D_{G\times\mathbb{C}^*}(\mathfrak{g})$.

2.3. Definition. *The graded Hecke algebra associated to G, L, and $\mathcal{L}$ is the algebra $H^*_{\mathcal{L}} = Ext^*_{D_{G\times\mathbb{C}^*}(\mathfrak{g})}(K_{\mathcal{L}}, K_{\mathcal{L}})$.*

We identify this algebra with Lusztig's definition as follows. Let $\ddot{\mathfrak{p}} \stackrel{\text{def}}{=} \dot{\mathfrak{p}} \underset{\mathfrak{g}}{\times} \dot{\mathfrak{p}}$ with first and second projections p and q to $\dot{\mathfrak{p}}$. Then the equivariant version of Lemma

4 in Appendix A shows that

$$H^*_{\mathcal{L}} = Ext^*_{D_{G\times\mathbb{C}^*}(\ddot{\mathfrak{p}})}(p^*\dot{\mathcal{L}}, q^!\dot{\mathcal{L}}).$$

Define similarly $\dot{\mathcal{C}} \overset{\text{def}}{=} G \underset{P}{\times} (\mathcal{C} + \mathfrak{n})$ and projections

$$\ddot{\mathcal{C}} \overset{\text{def}}{=} \dot{\mathcal{C}} \underset{\mathcal{N}}{\times} \dot{\mathcal{C}} \overset{p_0, q_0}{\longrightarrow} \dot{\mathcal{C}}.$$

Let $\dot{\mathcal{L}}_0$ be the local system on $\dot{\mathcal{C}}$ induced by $\mathcal{L}_0$. Then $\dot{\mathcal{L}}_0 = \dot{\mathcal{L}}|\dot{\mathcal{C}}$ and we also have the local system $\ddot{\mathcal{L}}_0 \overset{\text{def}}{=} R\mathcal{H}om(p_0^*\dot{\mathcal{L}}_0, q_0^*\dot{\mathcal{L}}_0)$ on $\ddot{\mathcal{C}}$. Since $j_!(\mathcal{L}_0) = j_*(\mathcal{L}_0)$ we have

$$\begin{aligned} H^*_{\mathcal{L}} &= Ext^*_{D_{G\times\mathbb{C}^*}(\ddot{\mathcal{C}})}(p_0^*\dot{\mathcal{L}}_0, q_0^!\dot{\mathcal{L}}_0) \\ &= H^*_{G\times\mathbb{C}^*}(\ddot{\mathcal{C}}, R\mathcal{H}om(p_0^*\dot{\mathcal{L}}_0, q_0^*\dot{\mathcal{L}}_0 \otimes \omega_{\dot{\mathcal{C}}}[-2\dim \dot{\mathcal{C}}])) \\ &= H_*^{G\times\mathbb{C}^*}(\ddot{\mathcal{C}}, \ddot{\mathcal{L}}_0). \end{aligned}$$

(use [Lu1, 3.2b, 3.3g, and the proof of 4.7]). This is essentially Lusztig's definition.

2.4 The subalgebra $H^0_{\mathcal{L}}$. Let $\mathfrak{t}$ be the Lie algebra of the torus $T = Z(L)_0$. Roots of $\mathfrak{t}$ in $\mathfrak{g}$ form a (possibly non–reduced) root system $\Delta \subset \mathfrak{t}^*$ and the group $W = N_G(T)/Z_G(T)$ is the Weyl group of Δ. The group W acts on $K_{\mathcal{L}}$ in a $(G \times \mathbb{C}^*)$–equivariant way. We recall the construction of this action. Let

$$\mathfrak{t}_{reg} = \{X \in \mathfrak{t} | Z_{\mathfrak{g}}(x) = \mathfrak{l}\}$$

The restriction of π to the map

$$G \underset{P}{\times} (\mathcal{C} + \mathfrak{t}_{reg} + \mathfrak{n}) \to \mathfrak{g}$$

is a principal W–bundle over its image. Let $\dot{\mathcal{L}} \boxtimes \mathbb{C}_{\mathfrak{t}_{reg}}$ and $\dot{\mathcal{L}} \boxtimes \mathbb{C}_{\mathfrak{t}}$ be the $G \times \mathbb{C}^*$ equivariant sheaves on $G \times_P \mathfrak{p}$ corresponding to the indicated sheaves on $\mathfrak{p}$.

Since $\mathcal{L}$ is W-equivariant (see below), this implies that W acts on

$$\pi_!(\dot{\mathcal{L}} \boxtimes \mathbb{C}_{\mathfrak{t}_{reg}})$$

and Lusztig has shown that its intersection cohomology extension is (up to a shift)

$$\pi_!(\dot{\mathcal{L}} \boxtimes \mathbb{C}_{\mathfrak{t}})$$

As a consequence,

$$\pi_!\dot{\mathcal{L}} = \pi_!(\dot{\mathcal{L}} \boxtimes \mathbb{C}_{\mathfrak{t}})|\mathcal{N},$$

has a W–action.

This gives a map of algebras

$$\mathbb{C}[W] \to Hom_{D_{G\times\mathbb{C}^*}(\mathcal{N})}(K_{\mathcal{L}}, K_{\mathcal{L}}) = H^0_{\mathcal{L}}$$

which is an isomorphism.

To see that $\mathcal{L}$ is W-equivariant, fix some $e \in \mathcal{C}$ and an $\mathfrak{sl}(2)$ triple e, h, f in $\mathfrak{l}$ and let $C = Z_G(e, h, f)_0$. Then T is a Cartan subgroup of C and $W = N_G(T)/L \cong N_C(T)/T$ [Lu1]. One easilly checks that $Z_L(e) \subset Z_G(e)$ gives an isomorphism of component groups, so that $\mathcal{L}_0$ extends to a G-equivariant local system $\mathcal{L}_1$ on the G-orbit ${}^G e$. Now $\mathcal{L}_0 = \mathcal{L}_1|\mathcal{C}$ is equivariant for $N_C(T) \subset G$ and thus also for $N_C(T)/T = W$.

2.5 The subalgebra $\mathcal{O}^*$. As above, let us fix some $e \in \mathcal{C}$ and an $\mathfrak{sl}(2)$ triple e, h, f in $\mathfrak{l}$. The corresponding map $\phi : SL(2) \to G$ defines a subgroup

$$\mathbb{C}^*{}_\phi = \{(\phi(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}), z) | z \in \mathbb{C}^*\} \subset G \times \mathbb{C}^*$$

and $\tilde{T} = T \cdot \mathbb{C}^*{}_\phi$ is a maximal connected reductive subgroup of the stabilizer $(L \times \mathbb{C}^*)_e$ of e in $L \times \mathbb{C}^*$. On the level of Lie algebras $\tilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathbb{C}_\phi$ for $\mathbb{C}_\phi = \mathbb{C} \cdot (h, 1) \subset \mathfrak{g} + \mathbb{C}$. $\tilde{T}$ appears in an exact sequence $0 \to T \to \tilde{T} \to \mathbb{C}^* \to 0$ defined via $T \subset \tilde{T} \subset L \times \mathbb{C}^* \to \mathbb{C}^*$. The graded algebra $\mathcal{O}^* = Ext^*_{D_{G\times\mathbb{C}^*}(\dot{\mathfrak{p}})}(\dot{\mathcal{L}}, \dot{\mathcal{L}})$ is canonically isomorphic to the equivariant cohomology $H^*{}_{\tilde{T}}(pt) = \mathbb{C}[\tilde{\mathfrak{t}}]$ ($\tilde{\mathfrak{t}}^*$ lies in degree 2). Functoriality of π_* gives an algebra map

$$\mathcal{O}^* \stackrel{\text{def}}{=} H^*_{\tilde{T}} \cong Ext^*_{D_{G\times\mathbb{C}^*}(\dot{\mathfrak{p}})}(\dot{\mathcal{L}}, \dot{\mathcal{L}}) \stackrel{\pi_*}{\to} Ext^*_{D_{G\times\mathbb{C}^*}(\mathcal{N})}(K_{\mathcal{L}}, K_{\mathcal{L}}) = H^*_{\mathcal{L}}$$

which is injective.

2.6 The algebra $H^*_{\mathcal{L}}$. Finally, multiplication in $H^*_{\mathcal{L}}$ gives an isomorphism of vector spaces $\mathbb{C}[\tilde{\mathfrak{t}}] \otimes \mathbb{C}[W] \to H^*_{\mathcal{L}}$. In order to describe the algebra structure on $H^*_{\mathcal{L}}$ completely, we recall the following commutation relation :

$$s_i \xi - s_i(\xi) s_i = c_i r \frac{(1-s_i)\xi}{\alpha_i}, \qquad \xi \in \mathbb{C}[\tilde{\mathfrak{t}}].$$

Here $\alpha_1, ..., \alpha_m$ are the simple roots of Δ corresponding to minimal parabolic subgroups P_i that contain P and the s_i are the corresponding simple reflections. The integer $c_i - 1$ is the smallest power e such that $(adx)^e \mathfrak{l}_i \cap \mathfrak{n} = 0$ for the Levi factor L_i of P_i which contains L. The functional $r \in \tilde{\mathfrak{t}}^* \subset \mathbb{C}[\tilde{\mathfrak{t}}]$ is the differential of the projection $\tilde{T} \to \mathbb{C}^*$.

2.7. In the case when $\mathfrak{l}$ is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and $\mathcal{L}_0$ is the constant sheaf supported at $0 \in \mathfrak{h}$, then if the equivariant structure is ignored, the induced sheaf is the usual Springer sheaf and the integers c_i all equal 2. In this setting, when the equivariant structure is used, the resulting algebra is the graded analogue of the affine Hecke algebra considered in [KL].

3. The action of Fourier transform on the graded Hecke algebra

3.1. Fourier transform [KS, 3.7]. For any finite dimensional vector space V, $D^b_c(V)$ contains the full subcategory $D_{con}(V)$ of conic sheaves, i.e., complexes of sheaves $\mathcal{A}$ on V such that all its cohomology sheaves are constant on the $\mathbb{R}^+$-orbits. Fourier transform $\mathcal{F} : D_{con}(V) \to D_{con}(V^*)$ is an equivalence of categories between conic sheaves on V and on the dual vector space V^*.

3.2. Fourier transform $\mathcal{F}_\mathfrak{g}$ of $G \times \mathbb{C}^*$–sheaves. We will fix a nondegenerate G-invariant bilinear form on $\mathfrak{g}$. Then the representation $\mathfrak{g}^*$ of $G \times \mathbb{C}^*$ dual to $\mathfrak{g}$ is isomorphic to $\mathfrak{g}$ as a vector space, but the new action of $G \times \mathbb{C}^*$ is related to the old one by the involution ϵ of $G \times \mathbb{C}^*$, $\epsilon(g, \lambda) = (g, \lambda^{-1})$. The involution ϵ gives an equivalence $D_{G\times\mathbb{C}^*}(\mathfrak{g}^*) \stackrel{\epsilon}{\cong} D_{G\times\mathbb{C}^*}(\mathfrak{g})$ and we define $\mathcal{F}_\mathfrak{g} : D_{G\times\mathbb{C}^*}(\mathfrak{g}) \to D_{G\times\mathbb{C}^*}(\mathfrak{g})$ as the composition $\epsilon \circ \mathcal{F}$ ($\mathbb{C}^*$-equivariant sheaves are certainly conic).

3.3. Nilpotent Fourier transform. Let $\mathcal{N} \overset{i}{\hookrightarrow} \mathfrak{g}$ be the nilpotent cone in $\mathfrak{g}$. The nilpotent Fourier transform $D_{G\times\mathbb{C}^*}(\mathcal{N}) \overset{\mathcal{F}_\mathcal{N}}{\to} D_{G\times\mathbb{C}^*}(\mathcal{N})$ is the composition $\mathcal{F}_\mathcal{N} = i^* \circ \mathcal{F}_\mathfrak{g} \circ i_!$.

3.4 Theorem [Lu6]. *If $\mathcal{L}$ is a cuspidal sheaf on $\mathcal{N}$ and $\mathfrak{g}$ is semisimple then $\mathcal{F}(\mathcal{L})$ is isomorphic to $\mathcal{L}$ ($\mathcal{F}$ is the ordinary Fourier trasform on $D_{con}(\mathfrak{g})$) (see [Mi] for a characteristic independent proof).*

3.5. Lemma. $\mathcal{F}_\mathcal{N} \circ Ind_L^G = Ind_L^G \circ \mathcal{F}_{\mathcal{N}\cap\mathfrak{l}}$.

Proof. It is easy to check that our Fourier transform commutes with the restriction functor Res_L^G which is adjoint to induction (see [Mi]).

3.6. Let $\mathcal{A}$ be a sheaf in $D_{G\times\mathbb{C}^*}(\mathcal{N})$. Any isomorphism $\phi : \mathcal{F}_\mathcal{N}(\mathcal{A}) \to \mathcal{A}[s]$ (with s an integer) defines an algebra automorphism I_ϕ of $Ext^*_{D_{G\times\mathbb{C}^*}(\mathcal{N})}(\mathcal{A},\mathcal{A})$:

$$Ext^*_{D_{G\times\mathbb{C}^*}(\mathcal{N})}(\mathcal{A},\mathcal{A}) \overset{\mathcal{F}_\mathcal{N}}{\to} Ext^*_{D_{G\times\mathbb{C}^*}(\mathcal{N})}(\mathcal{F}_\mathcal{N}\mathcal{A},\mathcal{F}_\mathcal{N}\mathcal{A}) \to$$
$$\to Ext^*_{D_{G\times\mathbb{C}^*}(\mathcal{N})}(\mathcal{A}[s],\mathcal{A}[s]) = Ext^*_{D_{G\times\mathbb{C}^*}(\mathcal{N})}(\mathcal{A},\mathcal{A}).$$

Obviously the automorphism depends only on $\mathbb{C}^*\cdot\phi$.

3.7. Now let $G, P, L, \mathcal{L}$ etc. be as in (2.1). By (3.4) and (1.1a) there is an isomorphism $\phi_\mathcal{L} : \mathcal{F}_{\mathcal{N}\cap\mathfrak{l}}(\mathcal{L}) \to \mathcal{L}$ in $D_{L\times\mathbb{C}^*}(\mathcal{N}\cap\mathfrak{l})$. By (3.5) $\phi = Ind_L^G(\phi_\mathcal{L})$ is an isomorphism $\mathcal{F}_\mathcal{N}(K_\mathcal{L}) \to K_\mathcal{L}$ in $D_{G\times\mathbb{C}^*}(\mathcal{N})$. Since $\mathcal{L}$ is a shifted irreducible perverse sheaf, $\phi_\mathcal{L}$ is unique up to a scalar. Thus the automorphism $\iota = I_\phi$ of the algebra $Ext^*_{D_{G\times\mathbb{C}^*}(\mathcal{N})}(K_\mathcal{L},K_\mathcal{L}) = H^*_\mathcal{L}$ is well defined.

3.8 Theorem.
(a) ι is the identity map on $\mathbb{C}[\mathfrak{t}]$.
(b) $\iota(r) = -r$.
(c) $\iota(w) = (-1)^{l(w)}w$, for $w \in W$ (cf. [HK], [Lu3]).

Proof. Recall that $\mathcal{O}^* \overset{i}{\subset} H^*_\mathcal{L}$ is given by

$$Ind_L^G : Ext^*_{D_{L\times\mathbb{C}^*}(\mathfrak{l})}(\mathcal{L},\mathcal{L}) \mapsto Ext^*_{D_{G\times\mathbb{C}^*}(\mathfrak{g})}(Ind_L^G\mathcal{L}, Ind_L^G\mathcal{L}).$$

Let $\iota_\mathfrak{l}$ be the automorphism $\iota_\mathfrak{l}$ of $\mathcal{O}^* = Ext^*_{D_{L\times\mathbb{C}^*}(\mathfrak{l})}(\mathcal{L},\mathcal{L})$ defined by the analogue of ι for $\mathfrak{l}$. It is easy to check that $i\circ\iota_\mathfrak{l} = \iota\circ i$ on $\mathcal{O}^*$. Since the canonical map $H^*_{L\times\mathbb{C}^*} \mapsto Ext^*_{D_{L\times\mathbb{C}^*}(\mathfrak{l})}(\mathcal{L},\mathcal{L}) = \mathcal{O}^*$ factors through $H^*_{\tilde{T}} \cong \mathcal{O}^*$ and T acts trivially on $\mathfrak{l}$ and on $\mathcal{L}$, we see that ι is trivial on $\mathbb{C}[\mathfrak{t}] \subset \mathcal{O}^*$ and $\iota(r) = -r$ due to the use of ϵ in the definition of $\iota_\mathfrak{l}$. This proves (a) and (b). (c) appears in the above references, but one may also use the equivariant structure to argue as follows. ι preserves $\mathbb{C}[W]$ since it is a map of graded algebras. Let s be any simple reflection in W. Then ι preserves $\mathbb{C}[s]$ since this is the subspace of $\mathbb{C}[W]$ which acts as the identity on the hyperplane perpindicular to α. It follows that $\iota(s) = s$ or $-s$. Let ξ be a weight such that $\xi(\check{\alpha}) = 1$. Then the relation given in 2.6 implies that $\iota(s) = -s$.

3.9. There are two additional involutions on $H^*_\mathcal{L}$. Define $\overline{IM}$ by the formulas

$$\overline{IM}(w) = (-1)^{l(w)}w,\ w \in W,\ \overline{IM}(r) = r,\ \overline{IM}(x) = -x,\ x \in \mathfrak{t}^*.$$

Define the involution κ by $\kappa(w) = w$, $w \in W$ and $\kappa(x) = -x$, $x \in \tilde{\mathfrak{t}}$. It is clear that $\iota = \kappa \circ \overline{IM}$.

4. Localization of $H^*_{\mathcal{L}}$

In this section, we explain how to use theorem 3.8 to prove our main theorem for graded Hecke algebras in 4.14. To do this, we relate the action of Fourier transform on $\mathfrak{g}$ on the graded Hecke algebra $H^*_{\mathcal{L}}$ with the action of Fourier transform on the fixed point set $\mathfrak{g}^\alpha$ on the specialization $H^*_{\mathcal{L},\alpha}$ at $\alpha \in \tilde{\mathfrak{t}}$. This is easy once we give a geometric interpretation of specialization in 4.10.

4.1. As in 2.5, fix some $e \in \mathcal{C}$ and an $\mathfrak{sl}_2$–triple e, h, f in $\phi(\mathfrak{sl}(2))$ in $\mathfrak{l}$. Then $C = Z_G(e,h,f)_0$ and $\tilde{C} = C \cdot {\mathbb{C}^*}_\phi$ are maximal connected reductive subgroups of $Z_G(e)$ and the stabilizer $(G \times \mathbb{C}^*)_e$ of e. Recall that T is a Cartan subgroup of C, $W = N_G(T)/L \cong N_C(T)/T \cong N_G(L)/L$ and similarly, $\tilde{T} = T \cdot {\mathbb{C}^*}_\phi$ is a Cartan subgroup of $\tilde{C}$ and $W \cong N_{\tilde{C}}(\tilde{T})/\tilde{T}$ [Lu1].

4.2 Lemma.
a) $Z(H^*_{\mathcal{L}}) = (\mathcal{O}^*)^W \cong H^*_{\tilde{C}}$.
b) The image of $H^*_{G\times\mathbb{C}^*} \to Ext_{D_{G\times\mathbb{C}^*}(\mathfrak{g})}(K_{\mathcal{L}}, K_{\mathcal{L}}) = H^*_{\mathcal{L}}$ *is precisely* $Z(H^*_{\mathcal{L}})$.

Proof. Part (a) is [Lu1, 6.5]. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ and let W_G be the Weyl group of G. The assertion is that $\mathbb{C}[\mathfrak{h}]^{W_G}$ surjects onto $\mathbb{C}[\mathfrak{t}]^W$. This is not true for every subgroup of the Weyl group of a simple group but is known in some cases (see [He]). We can assume G is simple and simply connected. In the case $e = 0$ and $\mathfrak{l}$ is abelian, the assertion is obvious. The remaining cases are listed in [Lu1, 2.13]. It is enough to check that the representation ring of the bigger group surjects onto the representation ring of the smaller group, as one sees by completing. For the classical groups the claim is easy to check because the embedding of $\mathfrak{t}$ in $\mathfrak{h}$ is easy to describe and one has a simple description of the representation rings. For example, for $\mathfrak{sl}(k)$ embedded diagonally inside $\mathfrak{sl}(nk)$ the claim follows because each of the first k fundamental representations of $\mathfrak{sl}(nk)$ restricts to a sum of n copies of the corresponding fundamental representation of $\mathfrak{sl}(k)$. The two exceptional cases give the embedding of G_2 inside of E_6 as the centralizer of an $\mathfrak{sl}(3)$ and the (unique) embedding of F_4 inside of E_7. The surjectivity of $R(G) \mapsto R(C)$ can be checked by examining the branching laws for basic representations in [MP].

4.3. The characters of $Z(H^*_{\mathcal{L}}) = H^*_{\tilde{C}} = \mathbb{C}[\tilde{\mathfrak{t}}/W]$ correspond to W–orbits in $\tilde{\mathfrak{t}}$. Thus, any $\alpha \in \tilde{\mathfrak{t}}$ defines a one-dimensional module $\mathbb{C}_\alpha$ for $Z(H^*_{\mathcal{L}})$, which is also a one-dimensional module for $H^*_{G\times\mathbb{C}^*}$ via the surjection $H^*_{G\times\mathbb{C}^*} \mapsto Z(H^*_{\mathcal{L}})$.

4.4 Localization of the algebra $H^*_{\mathcal{L}}$ at $\alpha \in \tilde{\mathfrak{t}}$. Fix $\alpha = (\sigma, r_0) \in \tilde{\mathfrak{t}} \subset (\mathfrak{t} + \mathbb{C}\cdot h) \oplus \mathbb{C}$ and let $S \subset \tilde{T}$ be the smallest torus whose Lie algebra contains α. Then the specialized algebra $H^*_{\mathcal{L},\alpha} \stackrel{\text{def}}{=} H^*_{\mathcal{L}} \otimes_{Z(H^*_{\mathcal{L}})} \mathbb{C}_\alpha$ equals $H^*_{\mathcal{L}} \otimes_{H^*_{G\times\mathbb{C}^*}} \mathbb{C}_\alpha$ by Lemma 4.2b.

4.5. In order to understand $H^*_{\mathcal{L},\alpha}$ geometrically, we begin by revising our description of $H^*_{\mathcal{L}}$. Let $\mathcal{C} + \mathfrak{n} \stackrel{\nu}{\hookrightarrow} \mathfrak{p}$ and $\dot{\mathcal{C}} \stackrel{\dot{\nu}}{\hookrightarrow} \dot{\mathfrak{p}}$ be the obvious inclusions and let $\tau = \pi\dot{\nu} : \dot{\mathcal{C}} \mapsto \mathfrak{g}$. Let ${(G \times \mathbb{C}^*)^\alpha}_0$ be the connected component of the centralizer of α in $G \times \mathbb{C}^*$ and let R be any subgroup of ${(G \times \mathbb{C}^*)^\alpha}_0$. We have,

$$\begin{aligned} H^*_{\mathcal{L},R} &\stackrel{\text{def}}{=} Ext^*_{D_R(\mathfrak{g})}(\pi_*\dot{\mathcal{L}}, \pi_*\dot{\mathcal{L}}) = Ext^*_{D_R(\mathfrak{g})}(\pi_!\dot{\mathcal{L}}, \pi_*\dot{\mathcal{L}}) \\ &= Ext^*_{D_R(\mathfrak{g})}(\pi_!\dot{\nu}_!\dot{\mathcal{L}}_0, \pi_*\dot{\nu}_*\dot{\mathcal{L}}_0) = Ext^*_{D_R(\mathfrak{g})}(\tau_!\dot{\mathcal{L}}_0, \tau_*\dot{\mathcal{L}}_0) \end{aligned}$$

since π is proper and $\mathcal{L}_0$ and thus also $\dot{\mathcal{L}}_0$ are clean local systems.

Consider the first and second projections p and q from $\ddot{\mathcal{C}}$ to $\dot{\mathcal{C}}$. Using base change for the Cartesian square

$$\begin{array}{ccc} \ddot{\mathcal{C}} & \xrightarrow{q} & \dot{\mathcal{C}} \\ p\downarrow & & \downarrow\tau \\ \dot{\mathcal{C}} & \xrightarrow{\tau} & \mathfrak{g} \end{array}$$

gives

$$\begin{aligned} H^*_{\mathcal{L},R} &= Ext^*_{D_R(\mathfrak{g})}(\tau_!\dot{\mathcal{L}}_0, \tau_*\dot{\mathcal{L}}_0) = Ext^*_{D_R(\mathfrak{g})}(\dot{\mathcal{L}}_0, \tau^!\tau_*\dot{\mathcal{L}}_0) \\ &= Ext^*_{D_R(\dot{\mathcal{C}})}(\dot{\mathcal{L}}_0, p_*q^!\dot{\mathcal{L}}_0) = Ext^*_{D_R(\ddot{\mathcal{C}})}(p^*\dot{\mathcal{L}}_0, q^!\dot{\mathcal{L}}_0) \\ &= H^*_R(\ddot{\mathcal{C}}, R\mathcal{H}om(p^*\dot{\mathcal{L}}_0, q^!\dot{\mathcal{L}}_0)). \end{aligned}$$

4.6. For a space Z with $\tilde{T}$ action, let Z^α denote the fixed point set of S. We have the inclusions of fixed point sets $\ddot{\rho} : \ddot{\mathcal{C}}^\alpha \hookrightarrow \ddot{\mathcal{C}}$ and $\dot{\rho} : \dot{\mathcal{C}}^\alpha \hookrightarrow \dot{\mathcal{C}}$. Let $\ddot{\mathcal{C}}_p = \{(x,y) \in \ddot{\mathcal{C}} | x \in \dot{\mathcal{C}}^\alpha\}$ and let $\ddot{\mathcal{C}}_q = \{(x,y) \in \ddot{\mathcal{C}} | y \in \dot{\mathcal{C}}^\alpha\}$.

We have a commutative diagram of Cartesian squares

$$\begin{array}{ccccc} \ddot{\mathcal{C}}^\alpha & \longrightarrow & \ddot{\mathcal{C}}_q & \longrightarrow & \dot{\mathcal{C}}^\alpha \\ \downarrow & & \downarrow & & \dot{\rho}\downarrow \\ \ddot{\mathcal{C}}_p & \longrightarrow & \ddot{\mathcal{C}} & \xrightarrow{q} & \dot{\mathcal{C}} \\ \downarrow & & p\downarrow & & \tau\downarrow \\ \dot{\mathcal{C}}^\alpha & \xrightarrow{\dot{\rho}} & \dot{\mathcal{C}} & \xrightarrow{\tau} & \mathfrak{g} \end{array}$$

4.7. In this subsection, we relate a certain localization morphism Φ on the space $\ddot{\mathcal{C}}$ to a map of Ext algebras on $\mathfrak{g}$. This allows us to apply the localization principle to Ext algebras.

The inclusion $\ddot{\rho} : \ddot{\mathcal{C}}^\alpha \hookrightarrow \ddot{\mathcal{C}}$ induces a canonical localization map for the sheaf $\mathcal{S} = R\mathcal{H}om(p^*\dot{\mathcal{L}}_0, q^!\dot{\mathcal{L}}_0)$ on $\ddot{\mathcal{C}}$,

$$\ddot{\rho}_!\ddot{\rho}^!\mathcal{S} \to \mathcal{S}.$$

We apply equivariant cohomology to this map to obtain the localization morphism

$$H^*_R(\ddot{\mathcal{C}}, \ddot{\rho}_!\ddot{\rho}^!R\mathcal{H}om(p^*\dot{\mathcal{L}}_0, q^!\dot{\mathcal{L}}_0)) \xrightarrow{\Phi} H^*_R(\ddot{\mathcal{C}}, R\mathcal{H}om(p^*\dot{\mathcal{L}}_0, q^!\dot{\mathcal{L}}_0)) \tag{4.7.1}$$

By using the identity

$$\ddot{\rho}_!\ddot{\rho}^!R\mathcal{H}om(U,V) \cong R\mathcal{H}om(\ddot{\rho}_*\ddot{\rho}^*U, \ddot{\rho}_!\ddot{\rho}^!V),$$

we can identify Φ with the map

$$Ext_{D_R(\ddot{\mathcal{C}})}(\ddot{\rho}_*\ddot{\rho}^*p^*\dot{\mathcal{L}}_0, \ddot{\rho}_!\ddot{\rho}^!q^!\dot{\mathcal{L}}_0) \overset{\Phi}{\mapsto} Ext_{D_R(\ddot{\mathcal{C}})}(p^*\dot{\mathcal{L}}_0, q^!\dot{\mathcal{L}}_0)$$

defined using the canonical morphisms $\ddot{\rho}_!\ddot{\rho}^!q^!\dot{\mathcal{L}}_0 \to q^!\dot{\mathcal{L}}_0$ and $p^*\dot{\mathcal{L}}_0 \to \ddot{\rho}_*\ddot{\rho}^*p^*\dot{\mathcal{L}}_0$. Next we use base change for the Cartesian squares in 4.6 to identify Φ with the map

$$Ext_{D_R(\ddot{\mathcal{C}})}(p^*\dot{\rho}_*\dot{\rho}^*\dot{\mathcal{L}}_0, q^!\dot{\rho}_!\dot{\rho}^!\dot{\mathcal{L}}_0) \xrightarrow{\Phi} Ext_{D_R(\ddot{\mathcal{C}})}(p^*\dot{\mathcal{L}}_0, q^!\dot{\mathcal{L}}_0)$$

defined using the canonical maps $p^*\dot{\mathcal{L}}_0 \mapsto p^*\dot{\rho}_*\dot{\rho}^*\dot{\mathcal{L}}_0$ and $q^!\dot{\rho}_!\dot{\rho}^!\dot{\mathcal{L}}_0 \mapsto q^!\dot{\mathcal{L}}_0$.

Finally, by retracing the calculation in 4.5, we can identify Φ with the map

$$A_R \stackrel{\text{def}}{=} Ext^*_{D_R(\mathfrak{g})}(\tau_!\dot{\rho}_*\dot{\rho}^*\dot{\mathcal{L}}_0, \tau_*\dot{\rho}_!\dot{\rho}^!\dot{\mathcal{L}}_0) \xrightarrow{\Phi} Ext^*_{D_R(\mathfrak{g})}(\tau_!\dot{\mathcal{L}}_0, \tau_*\dot{\mathcal{L}}_0) = H^*_{\mathcal{L},R}$$

defined using the canonical localization morphisms, which we will denote by

$$\dot{\rho}_!\dot{\rho}^!\dot{\mathcal{L}}_0 \xrightarrow{\zeta} \dot{\mathcal{L}}_0 \xrightarrow{\xi} \dot{\rho}_*\dot{\rho}^*\dot{\mathcal{L}}_0.$$

Moreover, on both sides we can replace $\tau_!$ with τ_* because $\dot{\mathcal{L}}_0$ and $\dot{\mathcal{L}}_0|\dot{\mathcal{C}}^{\alpha}$ are clean local systems (see 2.1 for $\dot{\mathcal{L}}_0$ and Lemma 4.11 for $\dot{\mathcal{L}}_0|\dot{\mathcal{C}}^{\alpha}$).

4.8. We show that the above map Φ is a map of algebras after some minor adjustment. Let $u \in A_R$. Then $\Phi(u) = \tau_*\zeta \circ u \circ \tau_*\xi$, so

$$\Phi(u)\Phi(v) = \tau_*\zeta \circ u \circ \tau_*\xi \circ \tau_*\zeta \circ v \circ \tau_*\xi = \Phi(u \circ \tau_*(\xi\zeta) \circ v).$$

So if we define the algebra structure on A_R by $u * v = u \circ \tau_*(\xi\zeta) \circ v$ then Φ is a map of algebras.

Moreover, we can also define an algebra map $A_R \overset{\Psi}{\mapsto} Ext^*_{D_R(\mathfrak{g})}(\tau_*\dot{\rho}_*\dot{\rho}^*\dot{\mathcal{L}}_0, \tau_*\dot{\rho}_*\dot{\rho}^*\dot{\mathcal{L}}_0)$ by $\Psi u = \tau_*(\xi\zeta) \circ u$ since $\Psi(u)\Psi(v) = \tau_*(\xi\zeta) \circ u \circ \tau_*(\xi\zeta) \circ v = \Psi(u * v)$. This gives the following lemma.

4.9 Lemma. *There are canonical morphisms of H^*_R algebras,*

$$Ext^*_{D_R(\mathfrak{g})}(\tau_*\dot{\mathcal{L}}_0, \tau_*\dot{\mathcal{L}}_0) \xleftarrow{\Phi} Ext^*_{D_R(\mathfrak{g})}(\tau_*\dot{\rho}_*\dot{\rho}^*\dot{\mathcal{L}}_0, \tau_*\dot{\rho}_!\dot{\rho}^!\dot{\mathcal{L}}_0) \xrightarrow{\Psi} Ext^*_{D_R(\mathfrak{g})}(\tau_*\dot{\rho}_*\dot{\rho}^*\dot{\mathcal{L}}_0, \tau_*\dot{\rho}_*\dot{\rho}^*\dot{\mathcal{L}}_0)$$

defined by $\Phi(u) = \tau_*(\zeta) \circ u \circ \tau_*(\xi)$, $\Psi(u) = \tau_*(\xi\zeta) \circ u$, for the canonical morphisms of sheaves on $\dot{\mathcal{C}}$

$$\dot{\rho}_!\dot{\rho}^!\dot{\mathcal{L}}_0 \overset{\zeta}{\mapsto} \dot{\mathcal{L}}_0 \overset{\xi}{\mapsto} \dot{\rho}_*\dot{\rho}^*\dot{\mathcal{L}}_0.$$

4.10 Theorem. *Let $H \times \mathbb{C}^*$ be a Cartan subgroup of $L \times \mathbb{C}^*$, let S the subtorus of $H \times \mathbb{C}^*$ generated by α, and let $\dot{\mathcal{C}}^{\alpha} \xrightarrow{\Upsilon} \mathfrak{g}^{\alpha}$ be the projection. Let $\dot{\mathcal{L}}_0^{\alpha}$ be the restriction of the local system $\dot{\mathcal{L}}_0$ to the fixed point set $\dot{\mathcal{C}}^{\alpha}$. Then there is a canonical isomorphism of $\mathbb{C}$-algebras*

$$H^*_{\mathcal{L},\alpha} \cong Ext^*_{D_{H\times\mathbb{C}^*/S}(\mathfrak{g}^{\alpha})}(\Upsilon_*\dot{\mathcal{L}}_0^{\alpha}, \Upsilon_*\dot{\mathcal{L}}_0^{\alpha}) \otimes_{H^*_{H\times\mathbb{C}^*/S}} \mathbb{C}_0.$$

Proof. Since $H \times \mathbb{C}^*$ is a Cartan of $G \times \mathbb{C}^*$ there is an isomorphism of bifunctors

$$H^*_{G\times\mathbb{C}^*}(-,-) \otimes_{H^*_{G\times\mathbb{C}^*}} H^*_{H\times\mathbb{C}^*} \xrightarrow{\cong} H^*_{H\times\mathbb{C}^*}(-,-).$$

This implies

$$H^*_{\mathcal{L},\alpha} = Ext^*_{D_{G\times\mathbb{C}^*}(\mathfrak{g})}(\tau_*\dot{\mathcal{L}}_0, \tau_*\dot{\mathcal{L}}_0) \otimes_{H^*_{G\times\mathbb{C}^*}} H^*_{H\times\mathbb{C}^*} \otimes_{H^*_{H\times\mathbb{C}^*}} \mathbb{C}_{\alpha}$$

$$\cong Ext^*_{D_{H\times\mathbb{C}^*}(\mathfrak{g})}(\tau_*\dot{\mathcal{L}}_0, \tau_*\dot{\mathcal{L}}_0) \otimes_{H^*_{H\times\mathbb{C}^*}} \mathbb{C}_{\alpha}.$$

Next, the localization principle in appendix B.2 shows that the map Φ defined for $R = H \times \mathbb{C}^*$ in 4.7.1 becomes an isomorphism of vector spaces after one applies the functor $-\otimes_{H^*_{H\times\mathbb{C}^*}} \mathbb{C}_\alpha$. Thus the reformulation of Φ that appears in Lemma 4.9 induces an isomorphism of algebras localized at α,

$$Ext^*_{D_{H\times\mathbb{C}^*}(\mathfrak{g})}(\tau_*\dot\rho_*\dot\rho^*\dot{\mathcal{L}}_0, \tau_*\dot\rho_!\dot\rho^!\dot{\mathcal{L}}_0)\otimes_{H^*_{H\times\mathbb{C}^*}} \mathbb{C}_\alpha \overset{\Phi}{\cong} Ext^*_{D_{H\times\mathbb{C}^*}(\mathfrak{g})}(\tau_*\dot{\mathcal{L}}_0, \tau_*\dot{\mathcal{L}}_0)\otimes_{H^*_{H\times\mathbb{C}^*}} \mathbb{C}_\alpha$$

We now show that the algebra morphism Ψ defined in 4.9 also induces an isomorphism after localization.

For any connected component C of $\dot{\mathcal{C}}^\alpha$ there is a canonical isomorphism $\dot\rho^!(\dot{\mathcal{L}}_0|C) \cong \dot{\mathcal{L}}_0|C[-2\delta_C]$, where δ_C is the codimension of C in $\dot{\mathcal{C}}$. The morphism $\xi\circ\zeta : \dot\rho_!\dot\rho^!\dot{\mathcal{L}}_0 \mapsto \dot\rho_*\dot\rho^*\dot{\mathcal{L}}_0$ restricted to a component C may then be identified with a morphism in

$$Hom_{D_{H\times\mathbb{C}^*}(C)}(\dot\rho^*\dot{\mathcal{L}}_0[-2\delta_C], \dot\rho^*\dot{\mathcal{L}}_0) = Ext^{2\delta_C}_{D_{H\times\mathbb{C}^*}(C)}(\dot\rho^*\dot{\mathcal{L}}_0, \dot\rho^*\dot{\mathcal{L}}_0).$$

which in turn may be identified with multiplication by the Euler class $e_C \in H^{2\delta_C}_{H\times\mathbb{C}^*}(C,\mathbb{C})$ of the embedding $C \hookrightarrow \dot{\mathcal{C}}$. Let $e_{\dot\rho} \in H^*_{H\times\mathbb{C}^*}(\dot{\mathcal{C}}^\alpha, \mathbb{C})$ be the sum of the Euler classes e_C corresponding to the distinct components. It follows that the map of algebras Ψ in 4.9 can be identified with multiplication by $\tau_*\dot\rho_*(e_{\dot\rho})$ on $Ext^*_{D_{H\times\mathbb{C}^*}(\mathfrak{g})}(\tau_*\dot\rho_*\dot\rho^*\dot{\mathcal{L}}_0, \tau_*\dot\rho_*\dot\rho^*\dot{\mathcal{L}}_0)$. Since $e_C(\alpha) \in \mathbb{C}^*$ for any connected component C of $\dot{\mathcal{C}}^\alpha$, Ψ becomes an isomorphism when one applies the functor $-\otimes_{H^*_{H\times\mathbb{C}^*}} \mathbb{C}_\alpha$. This gives an isomorphism of algebras

$$H^*_{\mathcal{L},\alpha} \cong Ext^*_{D_{H\times\mathbb{C}^*}(\mathfrak{g})}(\tau_*\dot\rho_*\dot\rho^*\dot{\mathcal{L}}_0, \tau_*\dot\rho_*\dot\rho^*\dot{\mathcal{L}}_0) \otimes_{H^*_{H\times\mathbb{C}^*}} \mathbb{C}_\alpha.$$

It is easy to identify this last algebra with the algebra

$$Ext^*_{D_{H\times\mathbb{C}^*}(\mathfrak{g}^\alpha)}(\Upsilon_*\dot{\mathcal{L}}_0^\alpha, \Upsilon_*\dot{\mathcal{L}}_0^\alpha) \otimes_{H^*_{H\times\mathbb{C}^*}} \mathbb{C}_\alpha.$$

Finally, since the $H\times\mathbb{C}^*$ structures on $\mathfrak{g}^\alpha$ and $\Upsilon_*\dot{\mathcal{L}}_0^\alpha$ factor through $(H\times\mathbb{C}^*)/S$ structures, the last algebra is isomorphic to

$$Ext^*_{D_{(H\times\mathbb{C}^*)/S}(\mathfrak{g}^\alpha)}(\Upsilon_*\dot{\mathcal{L}}_0^\alpha, \Upsilon_*\dot{\mathcal{L}}_0^\alpha) \otimes_{H^*_{(H\times\mathbb{C}^*)/S}} H^*_{H\times\mathbb{C}^*} \otimes_{H^*_{H\times\mathbb{C}^*}} \mathbb{C}_\alpha$$

$$= Ext^*_{D_{(H\times\mathbb{C}^*)/S}(\mathfrak{g}^\alpha)}(\Upsilon_*\dot{\mathcal{L}}_0^\alpha, \Upsilon_*\dot{\mathcal{L}}_0^\alpha) \otimes_{H^*_{(H\times\mathbb{C}^*)/S}} \mathbb{C}_0$$

(using the fact that α projects to 0 in $(\mathfrak{h}+\mathbb{C})/\mathfrak{s}$).

4.11 Lemma [Lu7]. *For any semisimple $\alpha \in \mathfrak{g}+\mathbb{C}$, the local system $\dot{\mathcal{L}}_0|\dot{\mathcal{C}}^\alpha$ is clean for the embedding $\dot{\mathcal{C}}^\alpha \subset \mathfrak{p}^\alpha$.*

Our proof was obtained independently from Lusztig's proof.

Proof. Let $\alpha = (\sigma, r_0) \in \mathfrak{g}+\mathbb{C}$. The fixed point set $\dot{\mathcal{C}}^\alpha$ lies over $\mathcal{P}^\alpha = \mathcal{P}^\sigma$, which is a finite union of $Z_G(\sigma)$-orbits isomorphic to partial flag varieties of $Z_G(\sigma)$. Proving the lemma reduces to proving the lemma over each $Z_G(\sigma)$-orbit Q, where the problem reduces by $Z_G(\sigma)$-equivariance to the analogous question over a single point $\mathfrak{q} \in Q$.

Thus, we can replace the embedding $\dot{\mathcal{C}} \subset \dot{\mathfrak{p}}$ and the local system $\dot{\mathcal{L}}_0$ with the embedding $\dot{\mathcal{C}}_{\mathfrak{q}} \subset \mathfrak{q}$ of the fiber at $\mathfrak{q} \in \mathcal{P}^{\sigma}$ and the local system $\dot{\mathcal{L}}_0|\dot{\mathcal{C}}_{\mathfrak{q}}$.

Now choose $g \in G$ so that $\mathfrak{q} = {}^g\mathfrak{p}$. Then $Ad(g^{-1})$ identifies the data $(\dot{\mathcal{C}}_{\mathfrak{q}} \hookrightarrow \mathfrak{q}, \dot{\mathcal{L}}_0|\dot{\mathcal{C}}_{\mathfrak{q}}, \alpha)$ with the data $(\mathcal{C} \oplus \mathfrak{u} \hookrightarrow \mathfrak{p}, \mathcal{L}_0 \boxtimes \mathbb{C}_{\mathfrak{u}}, \beta)$ for $\beta = Ad(g^{-1})\alpha = (\sigma_1, r_0) \in \mathfrak{p} + \mathbb{C}$. Moreover, after further conjugation by P we may assume $\sigma_1 \in \mathfrak{l}$. Then we have $(\mathcal{C} \oplus \mathfrak{u})^{\alpha_1} = \mathcal{C}^{\beta} \oplus \mathfrak{u}^{\beta}$ and $(\mathcal{L}_0 \boxtimes \mathbb{C}_{\mathfrak{u}})|(\mathcal{C} \oplus \mathfrak{u})^{\beta} = \mathcal{L}_0|\mathcal{C}^{\beta} \boxtimes \mathbb{C}_{\mathfrak{u}^{\beta}}$. This implies that it suffices to prove that for $\mathcal{L}_0$ a cuspidal local system on $\mathcal{C} \subset \mathfrak{l}$ and $\beta \in \mathfrak{l}$ semisimple, $\mathcal{L}_0$ restricts to a clean local system on $\mathfrak{l}^{\beta}$.

If $\mathcal{C}^{\beta}$ is not empty, we may assume $e \in \mathcal{C}^{\beta}$ by using L-equivariance. Thus, β lies in the Lie algebra $(\mathfrak{l}+\mathbb{C})_e$ of the stabilizer of $e \in L \times \mathbb{C}^*$. Thus, $\beta \in \mathfrak{t}+\mathbb{C}\cdot(\phi(h), 1) = \tilde{\mathfrak{t}}$ for some $\mathfrak{sl}(2)$-triple e, h, f in $\mathfrak{l}$ (see 4.1). It follows that β can be written in the form $(r_0 \cdot \phi(h) + \tau, r_0)$ with $\tau \in \mathfrak{t} = Z(\mathfrak{l})$ and we can assume that $\tau = 0$. If $r_0 = 0$ then $\beta = 0$ so $\mathcal{C}^{\beta} = \mathcal{C}$ and the claim reduces to the fact that $\mathcal{L}_0$ is clean. The remaining case where $\beta = (h, 1)$ requires more argument.

It suffices to show that for our cuspidal sheaf $\mathcal{L}$ on $\mathfrak{l}$, restriction to the subspace $\mathfrak{l}^h_k = \{x \in \mathfrak{l}, [h, x] = k \cdot x\}$ is clean for any integer k. Let $x \in \mathfrak{l}^h_k$ lie outside $\mathcal{C}$. Since $\mathcal{L}$ is clean $(\mathcal{L}|\mathfrak{l}^h_k)_x = \mathcal{L}_x$ is zero. It suffices to check that the !-stalk $j^!(\mathcal{L}|\mathfrak{l}^h_k)$ is zero, where j is the embedding $x \hookrightarrow \mathfrak{l}^h_k$. Since h is in the Lie algebra of $N_L(\mathbb{C} \cdot x)$, it lies in the Lie algebra of some maximal reductive subgroup of $N_L(\mathbb{C} \cdot x)$ and this subalgebra is neccesarily of the form $Z_{\mathfrak{l}}(x, h_0, y) + \mathbb{C} \cdot h_0$ for some sl_2-triple x, h_0, y containing x. Note that h and h_0 commute.

Recall that $S = [x + Z_{\mathfrak{l}}(y)] \cap \mathcal{N}$ is a normal slice to $L \cdot x$ in $\mathfrak{l}$ and that the intersection $S \cap \mathfrak{l}^h_k$ is a normal slice to $Z_L(h)_0 \cdot x$ in $\mathfrak{l}^h_k$ [Gi] (for the stratifications by orbits of L and $Z_L(h)_0$). Moreover, if $\psi : SL(2) \to L$ corresponds to the triple x, h_0, y then $\mathbb{C}^*$ acting via $\{(\psi(\begin{pmatrix} z^2 & 0 \\ 0 & z^{-2} \end{pmatrix}), z^k)|z \in \mathbb{C}^*\} \cong \mathbb{C}^*$ contracts S and $S \cap \mathfrak{l}^h_k$ to x. Therefore, $H^*_c(S, \mathcal{L})$ is up to a shift the !-stalk of $\mathcal{L}$ at x, and $H^*_c(S \cap \mathfrak{l}^h_k, \mathcal{L}|\mathfrak{l}^h_k)$ is up to a shift the !-stalk of $\mathcal{L}|\mathfrak{l}^h_k$ at x. Since $\mathcal{L}$ is clean, it follows that $H^*_c(S, \mathcal{L}) = 0$, so also $H^*_{c,\mathbb{C}^*}(S, \mathcal{L}) = 0$. The localization principle Appendix B.2 then implies that for $\zeta \in \mathbb{C}^* = Lie(\mathbb{C}^*) - \{0\}$,

$$H^*_{c,\mathbb{C}^*}(S, \mathcal{L}) \otimes_{H^*_{\mathbb{C}^*}} \mathbb{C}_{\zeta} \cong H^*_{c,\mathbb{C}^*}(S \cap \mathfrak{l}^h_k, \mathcal{L}|\mathfrak{l}^h_k) \otimes_{H^*_{\mathbb{C}^*}} \mathbb{C}_{\zeta},$$

so that $H^*_{c,\mathbb{C}^*}(S \cap \mathfrak{l}^h_k, \mathcal{L}|\mathfrak{l}^h_k) \otimes_{H^*_{\mathbb{C}^*}} \mathbb{C}_{\zeta}$ is also zero. But $\mathbb{C}^*$ acts trivially on $\mathcal{L}|S^{\mathbb{C}^*}$ since it may be identified with

$$[(\mathcal{C} \hookrightarrow \mathfrak{l})_! \mathcal{L}_0]|S^{\mathbb{C}^*} = (\mathcal{C} \cap S^{\mathbb{C}^*} \hookrightarrow S^{\mathbb{C}^*})_!(\mathcal{L}_0|\mathcal{C} \cap S^{\mathbb{C}^*}).$$

Thus, $H^*_{c,\mathbb{C}^*}(S^{\mathbb{C}^*}, \mathcal{L}) \cong H^*_c(S^{\mathbb{C}^*}, \mathcal{L}|S^{\mathbb{C}^*}) \otimes_{\mathbb{C}} H^*_{\mathbb{C}^*}$, and it follows that $0 = H^*_c(S^{\mathbb{C}^*}, \mathcal{L}|S^{\mathbb{C}^*}) = H^*_c(S \cap \mathfrak{l}^h_k, \mathcal{L}|\mathfrak{l}^h_k)$. $\square$

4.12 Theorem. *Regard $\dot{\mathcal{L}}_0^{\alpha}$ as an object in $D_{(H \times \mathbb{C}^*)/S}(\dot{\mathcal{C}}^{\alpha})$.*

(a) $\Upsilon_*(\dot{\mathcal{L}}_0^{\alpha})$ is a semisimple complex in $D_{(H \times \mathbb{C}^*)/S}(\mathfrak{g}^{\alpha})$, that is,

$$\Upsilon_*(\dot{\mathcal{L}}_0^{\alpha}) = \bigoplus_{\pi \in J_{\alpha}} L^*{}_{\pi} \otimes \pi,$$

where J_{α} is the set of all irreducible perverse sheaves that appear in $\Upsilon_*(\dot{\mathcal{L}}_0^{\alpha})$ and the $L^*{}_{\pi}$ are certain graded vector spaces.

(b) $\pi \mapsto L^*{}_\pi$ is a bijection from J_α to the set of irreducible modules for the algebra $H^*_{\mathcal{L},\alpha}$.

Proof. To prove (a), first write Υ for the composition $\dot{\mathcal{C}}^\alpha \overset{\mu}{\hookrightarrow} \dot{\mathfrak{p}}^\alpha \overset{\eta}{\to} \mathfrak{g}^\alpha$. Since $\dot{\mathcal{L}_0}^\alpha$ is clean for μ, $\mu_*(\dot{\mathcal{L}_0}^\alpha)$ is a semisimple complex on $\dot{\mathfrak{p}}$. Then since η is proper, $\Upsilon_* \dot{\mathcal{L}_0}^\alpha = \eta_* \mu_* \dot{\mathcal{L}_0}^\alpha$ is again a semisimple complex by the equivariant version of the decomposition theorem [BL, 5.3]. The proof of (b) follows as in Lemma A.2 since the algebra

$$H^*_{\mathcal{L},\alpha} \cong Ext^*_{D_{H\times\mathbb{C}^*/S}(\mathfrak{g}^\alpha)}(\Upsilon_* \dot{\mathcal{L}_0}^\alpha, \Upsilon_* \dot{\mathcal{L}_0}^\alpha) \otimes_{H^*_{H\times\mathbb{C}^*/S}} \mathbb{C}_0$$

is a finite dimensional graded algebra.

4.13 Remark. This gives a classification of irreducible modules for $H^*_{\mathcal{L}}$ by W-conjugacy classes of pairs (α, π) with $\alpha \in \tilde{\mathfrak{t}}$ and $\pi \in J_\alpha$. Observe that $\tilde{T}$ is a Cartan subgroup of $\tilde{C}$ (4.1) and W is the Weyl group of $\tilde{C}$ so that we could also give the classification in terms of $\tilde{C}$-conjugacy classes of pairs (α, π) with $\alpha \in \tilde{\mathfrak{c}}$ semisimple and $\pi \in J_\alpha$. For a more explict classification one should describe J_α, i.e. show that the π which occur in J_α are precisely the data described by Lusztig in [Lu5, section 16].

4.14 Theorem. *Let $\alpha = (\sigma, r_0) \in \tilde{\mathfrak{t}}$, $\check{\alpha} = (-\sigma, r_0)$, and let $\pi \in J_\alpha$. Denote the modules $L^*{}_\pi$ from 4.12 by $L_{\pi,\alpha}$. Then*

(a)

$$\iota(L_{\pi,\alpha}) = L_{\mathcal{F}_{\mathfrak{g}^\alpha}\pi, -\check{\alpha}}.$$

(b)

$$\kappa(L_{\pi,\alpha}) = L_{\pi,-\alpha}.$$

(c)

$$\overline{IM}(L_{\pi,\alpha}) = L_{\mathcal{F}_{\mathfrak{g}^\alpha}\pi, \check{\alpha}}.$$

Proof. Recall that the isomorphism $H^*_{\mathcal{L},\alpha} \cong Ext^*_{D_{(H\times\mathbb{C}^*)/S}(\mathfrak{g}^\alpha)}(\Upsilon_* \dot{\mathcal{L}_0}^\alpha, \Upsilon_* \dot{\mathcal{L}_0}^\alpha) \otimes_{H^*_{(H\times\mathbb{C}^*)/S}} \mathbb{C}_0$ was obtained by applying the functor $\text{-} \otimes_{H_{H\times\mathbb{C}^*}} \mathbb{C}_\alpha$ to the algebra morphisms Φ and Ψ from Lemma 4.9. We rewrite the maps in Lemma 4.9 as

$$Ext^*_{D_R(\mathfrak{g})}(\pi_* \dot{\mathcal{L}}, \pi_* \dot{\mathcal{L}}) \overset{\Phi}{\leftarrow} Ext^*_{D_R(\mathfrak{g})}(\pi_* \dot{\rho}_* \dot{\rho}^* \dot{\mathcal{L}}, \pi_* \dot{\rho}_! \dot{\rho}^! \dot{\mathcal{L}}) \overset{\Psi}{\to} Ext^*_{D_R(\mathfrak{g})}(\pi_* \dot{\rho}_* \dot{\rho}^* \dot{\mathcal{L}}, \pi_* \dot{\rho}_* \dot{\rho}^* \dot{\mathcal{L}})$$

for $\dot{\mathfrak{p}}^\alpha \overset{\rho}{\hookrightarrow} \dot{\mathfrak{p}}$ and $R = H \times \mathbb{C}^*$. Here we write $\Phi(u) = \pi_*(\zeta) \circ u \circ \pi_*(\xi)$ and $\Psi(u) = \pi_*(\xi\zeta) \circ u$ for the canonical maps $\dot{\rho}_! \dot{\rho}^! \dot{\mathcal{L}} \overset{\zeta}{\mapsto} \dot{\mathcal{L}} \overset{\xi}{\mapsto} \dot{\rho}_* \dot{\rho}^* \dot{\mathcal{L}}$. Now (a) follows from naturality properties of Fourier transform, the obvious relation between $\mathcal{F}_\mathfrak{g} \rho_* \rho^*$ and $\mathcal{F}_{\mathfrak{g}^\alpha}$ ([KS, 3.7.14]), and the description of irreducible constituents in Theorem 4.12. For (b), observe that we have an obvious identification between $\Upsilon_*(\dot{\mathcal{L}_0}^\alpha)$ and $\Upsilon_*(\dot{\mathcal{L}_0}^{-\alpha})$ and similar identifications for all objects with superscript α appearing in 4.10. It follows that $H^*_{\mathcal{L},\alpha}$ is canonically isomorphic to $H^*_{\mathcal{L},-\alpha}$ and that the canonical isomorphism is given by κ. The second claim now follows from Theorem 4.12 and the third claim follows from (a), (b) and 3.8.

4.15 Remark. If one thinks of $H^*_{\mathcal{L}}$ (up to a shift) as $H_*^{G\times\mathbb{C}^*}(\ddot{\mathcal{C}}, \ddot{\mathcal{L}}_0)$ (see 2.3), then the localization result B.7 says that $H^*_{\mathcal{L},\alpha} \cong \Pi_*^{(H\times\mathbb{C}^*)/S}(\ddot{\mathcal{C}}^\alpha, \ddot{\mathcal{L}}_0|\ddot{\mathcal{C}}^\alpha)$ (the $(H \times \mathbb{C}^*)/S$-primitive part of the Borel-Moore homology). Theorem 4.10 above is a reinterpretation in terms of Ext-algebras.

5. Lifting to the affine Hecke algebra

In this section, we explain how to lift our results from the graded Hecke algebra to the affine Hecke algebra. This involves nothing more than recalling the equivalence between representations of the graded Hecke algebra and representations of the affine Hecke algebra and showing that the graded Iwahori-Matsumoto involution corresponds to the Iwahori-Matsumoto involution under this equivalence. Our main technical tool is a Hecke algebra version of a theorem of Casselman for p-adic groups, which asserts that affine Hecke algebra representations are distinguished by their weights (Theorem 5.5). For reasons of convenience, we use a variation on the Lusztig equivalence [Lu2] between graded Hecke algebra representations and affine Hecke algebra representations due to Barbasch and Moy [BM].

5.1 Affine Hecke algebras. The class of affine Hecke algebras we study is as follows. Let G be a connected reductive complex group with simply connected derived group. Let $M_0 = T \times \mathbb{C}^*$ be a maximal torus of $G \times \mathbb{C}^*$. $R(M_0) \cong R(T) \otimes \mathbb{C}[v, v^{-1}]$ and we will regard characters of T as elements of $R(M_0)$ via the embedding $\theta \mapsto \theta \otimes 1$. Let $\alpha_1, ..., \alpha_m$ be a set of simple roots. Then the affine Hecke algebra $\mathcal{H}$ is the algebra generated by $T_1, \ldots, T_m$ and $R(M_0)$ subject to relations

$$\begin{aligned} &(T_i + 1)(T_i - v^2) = 0 \\ &T_i T_j T_i \cdots = T_j T_i T_j \cdots, i \neq j \\ &f(T_i + 1) - (T_i + 1)s_i(f) = (f - s_i(f))\frac{\theta_\alpha v^2 - 1}{\theta_\alpha - 1} \end{aligned}$$

where $f \in R(M_0)$, θ_α is the character of T given by the root α, and the number of factors on each side in the second relation is given by the order of $s_i s_j$ in the Weyl group. This affine Hecke algebra is associated to root data $(X, R, Y, \check{R}, \Pi)$ and there is a corresponding positive system of roots. This determines a dominant set of weights for the characters in the span of R. We remark that the affine Hecke algebra $\mathcal{H}_L$ of a Levi factor L may be regarded as a subalgebra of $\mathcal{H}$ in the obvious way. The Iwahori-Matsumoto involution $IM : \mathcal{H} \to \mathcal{H}$ is the unique $\mathbb{C}[v, v^{-1}]$ algebra automorphism such that $IM(T_i) = -v^2(T_i)^{-1}$ and $IM(\theta) = \theta^{-1}$ for θ a character of T. IM induces an involution, also denoted IM, on representations.

More generally, one can consider affine Hecke algebras corresponding to groups without simply connected derived group and to cuspidal local systems $\mathcal{L}$ on Levi factors as in 2.5. This leads to certain complications in the relations (see [Lu2, 3.6]). Our main result of this section (5.8) applies to these settings also by an explicit computation (see Remark 5.11).

5.2 Assumptions on affine Hecke algebras. We assume that v acts by a nonzero positive real number v_0 not equal to 1. This assumption is not really necessary but simplifies the discussion. We set $q = {v_0}^2$.

5.3 Representation theory of affine Hecke algebras. We recall some standard facts about the representation theory of $\mathcal{H}$ (cf. [KL, section 7]). Let $s \in G$ be semisimple and let $\mathfrak{g}(s, v_0)$ be the ${v_0}^2 = q$ eigenspace of s in $\mathfrak{g}$. For $N \in \mathfrak{g}(s, v_0)$, let $A(s, N)$ be the component group of $Z_G(s) \cap Z_G(N)$. Let X_N^s be the joint fixed point set of s and $exp(N)$ in the flag variety X of G. Then $A(s, N)$ acts on $H_*(X_N^s)$. Kazhdan and Lusztig classified irreducible representations of $\mathcal{H}$ as G-conjugacy classes of triples (s, N, ρ) where $s \in G$ is semisimple, $N \in \mathfrak{g}(s, v_0)$, and ρ is an irreducible representation of $A(s, N)$ appearing in $H_*(X_N^s)$ [KL]. We denote the corresponding irreducible representation by $\mathcal{L}(s, N, \rho)$.

Let $V = \mathcal{L}(s, N, \rho)$ be an irreducible $\mathcal{H}$ module. The center $\mathcal{Z} = R(M_0)^W$ of $\mathcal{H}$ acts on V by a homomorphism $\chi : \mathcal{Z} \to \mathbb{C}$. χ is called the central character of V and χ is given by $f \mapsto f(s, v_0)$. We write $\chi = (W \cdot s, v_0)$. As an $R(T)$ module, V has a generalized weight space decomposition $V = \oplus_{x \in T} V_x$, where $V_x = \{v \in V | (\theta - \theta(x))^n \cdot v = 0, \theta \in R(T), n = dim(V)\}$. Hence, if $V_x \neq 0$, then $x \in W \cdot s$. If $V_x \neq 0$, x is called a weight of V. In particular, if V and V' have the same weights, then their central characters coincide. It is clear from the definition of IM that $IM(V_x) = IM(V)_{x^{-1}}$. It follows that if the weights of V are $\{x_1, \ldots, x_k\}$, then the weights of $IM(V)$ are $\{{x_1}^{-1}, \ldots, {x_k}^{-1}\}$. The representation V is called essentially tempered if $|\theta(x)| \leq 1$ for all dominant characters θ of T and all weights x of V [KL, 8.1]. If both V and $IM(V)$ are tempered, then $|\theta(x)| = 1$ for every weight x of V and dominant character θ of T. This implies that x is elliptic, so s is also elliptic since it is W-conjugate to x. Hence, $\mathfrak{g}(s, v_0) = \{0\}$, which implies there is a unique irreducible representation with central character $(W \cdot s, v_0)$.

The classification of irreducible representations can be reduced to the classification of tempered representations. Given s and N, we attach a parabolic $P(s, N)$ as follows [KL, 7.1]. We denote a diagonal matrix in $SL(2, \mathbb{C})$ with entries (t, t^{-1}) by $D(t)$. Choose a homomorphism $\phi : SL(2, \mathbb{C}) \to G$ such that $\phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = exp(N)$ and $s_1 = s\phi(D({v_0}^{-1}))$ commutes with the image of ϕ. Let $\mathfrak{g} = \oplus \mathfrak{g}_\alpha$ be the eigenspace decomposition of s_1.

Then $\mathfrak{p}(s, N) = \oplus_{|\alpha| \leq 1} \mathfrak{g}^\alpha$ is a parabolic subalgebra with Levi subalgebra $\mathfrak{l}(s, N) = \oplus_{|\alpha|=1} \mathfrak{g}_\alpha$. Let $P(s, N)$ and $L(s, N)$ be the corresponding connected complex groups. It follows that $s, s_1 \in L$ and $N \in \mathfrak{l}$. Moreover, the component group $A(s, N)$ is the same whether defined in $G, P(s, N)$, or $L(s, N)$ [KL,7.2a]. Let $\hat{\mathcal{L}}(s, N, \rho)$ be the irreducible representation of $\mathcal{H}_L$ determined by the above classification. Then $\mathcal{H} \otimes_{\mathcal{H}_L} \hat{\mathcal{L}}(s, N, \rho)$ has unique irreducible quotient $\mathcal{L}(s, N, \rho)$ [KL, 7.12]. We call this quotient the Langlands quotient. Moreover, $\mathcal{L}(s, N, \rho)$ is tempered if and only if $P(s, N) = G$, and if this is so, N is in the open orbit of $Z_G(s)$ on $\mathfrak{g}(s, v_0)$ [KL, 8.2 and proof of 7.12]. In particular, $\hat{\mathcal{L}}(s, N, \rho)$ is a tempered representation of $\mathcal{H}_L$.

We will need one further technical result.

5.4 Lemma. *Suppose $\mathcal{L}(s, N, \rho)$ and $\mathcal{L}(s', N', \rho')$ have the same weights. Then $P(s, N)$ is conjugate to $P(s', N')$ and $\hat{\mathcal{L}}(s, N, \rho)$ and $\hat{\mathcal{L}}(s', N', \rho')$ have the same weights.*

Proof. This is essentially the same as the proof of Proposition 7.11 in [KL], which asserts that if $\mathcal{L}(s, N, \rho) \cong \mathcal{L}(s', N', \rho')$ as $\mathcal{H}$-modules, then $P(s, N)$ is conjugate to $P(s', N')$ and $\hat{\mathcal{L}}(s, N, \rho) \cong \hat{\mathcal{L}}(s', N', \rho')$ as $\mathcal{H}_L$-modules. Indeed, only the weight structure is used in the proof that $P(s, N)$ is conjugate to $P(s', N')$. Moreover, $\hat{\mathcal{L}}(s, N, \rho)$ embeds in $\mathcal{L}(s, N, \rho)$ and the weights in $\hat{\mathcal{L}}(s, N, \rho)$ are characterized as the weights of $\mathcal{L}(s, N, \rho)$ on which a certain function t (depending only on the weights) vanishes. The second assertion follows immediately.

We now prove an affine Hecke algebra version of a theorem of Casselman [Ca].

5.5 Theorem. *Let V and V' be irreducible $\mathcal{H}$-modules. If V and V' have the same weights, then $V \cong V'$ as $\mathcal{H}$-modules.*

Proof. Let $V = \mathcal{L}(s, N, \rho)$ and $V' = \mathcal{L}(s', N', \rho')$. Since the weights of the two representations coincide, their central characters coincide, so we may assume $s = s'$. If we assume the theorem is known for all proper Levi subgroups of G, then by Lemma 5.4 and the fact that the Langlands quotient is unique, the theorem follows for all nontempered representations of $\mathcal{H}$. Thus we may assume that V is tempered. Since V and V' have the same weights, $IM(V)$ and $IM(V')$ have the same weights. If $IM(V)$ is also tempered, then we saw in 5.3 that there is a unique irreducible representation with central character $(W \cdot s, v_0)$. In this case, the theorem is obvious. If $IM(V)$ is not tempered, then our inductive assumption implies that $IM(V) \cong IM(V')$ as $\mathcal{H}$-modules, so that $V \cong V'$ as $\mathcal{H}$-modules.

5.6 Comparison between graded and affine Hecke algebras. We recall the equivalence between affine Hecke algebra representations and graded Hecke algebra representations following [BM] and show that the Iwahori-Matsumoto involution corresponds to the graded Iwahori-Matsumoto involution under this equivalence.

Let $\chi : \mathcal{Z} \to \mathbb{C}$ be a nonzero ring homomorphism with kernel $\mathcal{I}_\chi$ and let $\mathcal{H}_\chi = \mathcal{H}/\mathcal{H}\mathcal{I}_\chi$. Choose a representative (s, v_0) for χ and let $s = \sigma \cdot exp(h)$ be its polar decomposition, so that σ is elliptic and $exp(h)$ is hyperbolic. σ determines a root system R_σ given by the roots which are trivial on σ. Since we are assuming G has simply connected derived group, this root system is the set of roots for the connected group $Z_G(\sigma)$ ([SS, p. 197], where the fact is only stated when G is simply connected, but can be easily generalized). More precisely, if G has root datum $(X, R, Y, \check{R}, \Pi)$, then $Z_G(\sigma)$ has root datum $(X, R_\sigma, Y, \check{R}_\sigma, \Pi_\sigma)$. Here Π_σ is the set of positive indecomposable roots in R_σ and $\check{R}_\sigma$ is the image of R_σ under the bijection $R \to \check{R}$. Let W_σ be the corresponding Weyl group. This root system determines a graded Hecke algebra H_σ. Let $\mathfrak{m}$ be the Lie algebra of M_0 with $\mathbb{C}[\mathfrak{m}]$ and $\mathbb{C}[\hat{\mathfrak{m}}]$ its algebras of polynomial and holomorphic functions. The center of $H_\sigma = \mathbb{C}[\mathfrak{m}]^{W_\sigma}$, the W_σ invariant functions on $\mathfrak{m}$. This implies that an irreducible representation of the center of H_σ is determined by a W_σ orbit in $\mathfrak{m}$. We let $\overline{\chi}$ be the central character for H_σ given by the W_σ orbit $W_\sigma(h, r_0)$. We denote by $H_{\sigma,\overline{\chi}}$ the quotient of H_σ by the ideal generated by the character $\overline{\chi}$ of the center. We have the completion $\hat{H}_\sigma = H_\sigma \otimes_{\mathbb{C}[\mathfrak{m}]} \mathbb{C}[\hat{\mathfrak{m}}]$ of H_σ and we can extend $\overline{\chi}$ to a character of the center of $\hat{H}_\sigma$. We denote the quotient of $\hat{H}_\sigma$ by the ideal generated by $\overline{\chi}$ by $\hat{H}_{\sigma,\overline{\chi}}$. It is isomorphic to $H_{\sigma,\overline{\chi}}$. Let $n = |W \cdot \sigma|$ and let $\sigma = \sigma_1, \ldots, \sigma_n$ be the elements of $W \cdot \sigma$.

5.7 Theorem [BM, 4.3, 3.3(ii)]. *There is an isomorphism $\Phi_\sigma : \mathcal{H}_\chi \cong M_n(\hat{H}_{\sigma,\overline{\chi}})$, the $n \times n$ matrices with coefficients in $\hat{H}_{\sigma,\overline{\chi}}$. Under this isomorphism, the character θ_λ of T maps to the diagonal matrix with i^{th} entry, $\theta_\lambda(\sigma_i)e^\lambda$.*

Observe that the Iwahori-Matsumoto involution maps the central character $\chi = W \cdot (s, v_0)$ of $\mathcal{H}$ to the central character $\check{\chi} = W \cdot (s^{-1}, v_0)$. Similarly, the graded Iwahori-Matsumoto involution maps the central character $\overline{\chi} = (W_\sigma \cdot h, r_0)$ of H_σ to the central character $\check{\overline{\chi}} = (W_\sigma \cdot (-h), r_0)$.

As a consequence, there are induced algebra isomorphisms $IM : \mathcal{H}_\chi \to \mathcal{H}_{\check{\chi}}$ and $\overline{IM} : H_{\sigma,\overline{\chi}} \to H_{\sigma^{-1},\check{\overline{\chi}}}$. Recall that there is a natural Morita equivalence between representations of an algebra R and representations of the matrix algebra $M_n(R)$.

5.8 Theorem. *The Iwahori-Matsumoto involution corresponds to the graded Iwahori-*

Matsumoto involution under the above equivalences, i.e., the diagram

$$\begin{array}{ccc} Irr(\mathcal{H}_\chi) & \xrightarrow{IM} & Irr(\mathcal{H}_{\check{\chi}}) \\ \downarrow & & \downarrow \\ Irr(H_{\sigma,\overline{\chi}}) & \xrightarrow{\overline{IM}} & Irr(H_{\sigma^{-1},\overline{\check{\chi}}}) \end{array}$$

commutes, where the vertical arrows are induced by Φ_σ and $\Phi_{\sigma^{-1}}$, the equivalence between H_σ and $\hat{H}_\sigma$ modules, and Morita equivalence.

Proof. The identity $\Phi_{\sigma^{-1}} \circ IM(\theta_\lambda) = \overline{IM} \circ \Phi_\sigma(\theta_\lambda)$ is obvious from the definitions. It follows immediately that if V is an irreducible representation of $M_n(H_{\mathcal{O}_{\overline{\chi}}})$, then the weights of $\overline{IM}(\Phi_\sigma(V))$ are the same as the weights of $\Phi_{\sigma^{-1}}(IM(V))$. Since both are irreducible representations of $\mathcal{H}_\chi$, they are isomorphic by theorem 5.5. The result now follows from the equivalence between representations of R and of $M_n(R)$ $\square$.

Affine Hecke algebra representations can be classified via the equivalence with graded Hecke algebra representations and the geometric classification of graded Hecke algebra representations. Since the graded Iwahori-Matsumoto involution corresponds to modified Fourier transform by Theorem 4.14, the previous result implies the following theorem.

5.9 Theorem. *Assume v_0 is strictly positive and not equal to one. The Iwahori-Matsumoto involution and modified Fourier transform induce the same involution on irreducible affine Hecke algebra representations under the above geometric classification of affine Hecke algebra representations.*

5.10 Remark. *It is more standard to classify irreducible affine Hecke algebra representations using equivariant K-homology as in [KL]. In the next section, we will verify that the classification in [KL] is the IM shift of the classification described above. This implies that 5.9 is valid for the Kazhdan-Lusztig classification.*

5.11 Remark. *This theorem can also be proved by explicit computation. One finds that IM and $\overline{IM}$ commute with the equivalences up to conjugation by a unit of $\hat{A}$. This unit also appears in [BM, 5.7]. This technique shows that theorem 5.8 is true in the more general setting where $\mathcal{H}_{\mathcal{L}}$ is the affine Hecke algebra corresponding to any cuspidal local system on a Levi factor and where G need not have simply connected derived group. We omit the details.*

6. Comparison between the two classifications

In the last section, we observed that at fixed central characters, the affine Hecke algebra is isomorphic to a matrix algebra over the graded Hecke algebra (5.7). It follows immediately that their irreducible representations coincide. In [KL], a geometric classification of affine Hecke algebra representations is given and in [Lu1], a geometric classification of graded Hecke algebra representations is given. In this section, we observe that the sets parametrizing these two classifications are identical, and further show that up to some Iwahori-Matsumoto shift, the isomorphism of 5.7 identifies parameters. This result is undoubtedly well-known to the experts. Our proof consists in showing that after the IM shift, the two representations have the same weights, so that by 5.5 they are isomorphic. As we mentioned above, this implies our main theorem (0.1).

We retain the assumptions of the previous section (cf. 5.1 and 5.2).

6.1 Normalizing choices of simple roots. As we have remarked, the construction of irreducible affine Hecke algebra representations corresponds to the construction of irreducible graded Hecke algebra representations under the isomorphisms of the last section only after a shift. The reason for the discrepancy is that simple roots correspond to different line bundles in the two constructions. More precisely, let α be a simple root, let X_α be the variety of parabolics of type α, and let $\pi : X \to X_\alpha$ be the projection with tangent bundle to the fiber L_π. In [KL], L_π corresponds to the simple root α while in [Lu1], the dual line bundle ${L_\pi}^*$ corresponds to the simple root α. We will choose the convention from [Lu1]. This choice forces a change in the affine Hecke algebra action which corresponds to taking the Iwahori-Matsumoto shift of the action of the operators T_s given in [KL]. Given our convention, the action of $R(T)$ can be computed in the same way as in [KL].

6.2 Weights of affine Hecke algebra representations. We need some observations about weights of affine Hecke algebra representations (cf. [KL, 7.6]). Let M' be the smallest algebraic subgroup of M_0 containing (s, v_0). There is an algebra homomorphism $R(M') \to \mathbb{C}_{s,v_0}$ given by evaluation at (s, v_0). For a $R(M')$ module V, let $V_{s,v_0} = V \otimes_{R(M')} \mathbb{C}_{s,v_0}$. Then $K_0^{M'}(X)_{s,v_0} = E$ is an affine Hecke algebra module (via the IM shift of the structure given in [KL, section 3], as in [Lu4]). The commutative algebra $R(T)$ acts on E, which induces a generalized weight space decomposition $E = \oplus_{t \in W \cdot s} E_t$, where E_t is the subspace on which any character $\theta \in R(T)$ acts by $\theta(t)$ times a unipotent transformation.

6.3 Geometric description of weights. We give a geometric description of this decomposition. A character $\theta_\lambda \in R(T)$ correspond to a G-equivariant line bundle $L(\lambda)$ on X in the obvious way. Given a line bundle L on X and a semisimple element s in a Borel $B \in X$, $\psi_L^B(s)$ is defined by the action of s on the fiber of L over $B \in X$.

The fixed point set X^s for the semisimple element s is a union of components, $X = \sqcup X^{s,\alpha}$. Each component $X^{s,\alpha}$ is a homogeneous space for $Z_G(s)$ (isomorphic to its flag manifold). This induces a direct sum decomposition

$$K_0^{M'}(X^s)_{s,v_0} = \oplus K_0^{M'}(X^{s,\alpha})_{s,v_0}$$

and θ_λ acts on $K_0^{M'}(X^{s,\alpha})_{s,v_0}$ by $\psi^B_{L(\lambda)}(s)$ times a unipotent transformation, where B is any Borel in $X^{s,\alpha}$. It follows that $\psi^B_{L(\lambda)}(s) = \theta_\lambda(t)$ for some $t \in W \cdot s$. The components $X^{s,\alpha}$ then correspond to elements in $W \cdot s$ via the rule : $X^{s,t}$ is the component on which $R(T)$ acts by the generalized weight t.

Note that the above discussion yields a $Z_G(s)$ equivariant embedding of the flag manifold of $Z_G(s)$ into X as the component $X^{s,s}$ of X^s. More generally, let $s = \sigma \cdot exp(h)$ be the polar decomposition of s and let $\tilde{X}$ be the flag manifold of $Z_G(\sigma)$. Then $\tilde{X}^s$ may be embedded into X as the subset of X^s given by the components $X^{s,t}$ of X^s where t has elliptic part σ.

We now consider a nilpotent $N \in \mathfrak{g}(s, v_0)$. X_N^s is a union of connected components $X_N^{s,\beta}$, and $K_0^{M'}(X_N^s)_{s,v_0} = \oplus K_0^{M'}(X_N^{s,\beta})_{s,v_0}$. The generalized weight on $K_0^{M'}(X_N^{s,\beta})_{s,v_0}$ is then given by $t \in W \cdot s$, where $X^{s,t}$ is the unique connected component of X^s containing $X_N^{s,\beta}$. Moreover, $\tilde{X}_N^s$ is a union of components of X_N^s, and it follows from the above discussion that $K_0^{M'}(\tilde{X}_N^s)_{s,v_0}$ is the sum of the generalized weight spaces of $K_0^{M'}(X_N^s)_{s,v_0}$ for which the generalized weight has elliptic part σ.

We will denote projection onto this factor by E_σ. It is the same as the element denoted E_σ in [BM, 3.6]. In terms of the isomorphism $\mathcal{H}_\chi \cong M_n(\hat{H}_{\sigma,\overline{\chi}})$, E_σ corresponds to projection onto the first factor (see 5.7). In particular, $E_\sigma\theta_\lambda$ maps to $\theta_\lambda(\sigma)e^\lambda$ in $\hat{\mathbb{C}[\mathfrak{t}]}$.

6.4 Weights for graded Hecke algebra representations. A similar discussion applies to the graded Hecke algebra H_σ and the weights of $\mathbb{C}[\mathfrak{t}]$. Let $\tilde{M}'$ be the smallest algebraic subgroup of M containing $(exp(h), v_0)$ and if V is a $\mathbb{C}[\mathfrak{t}]$–module, let $V_{h,r_0} = V \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}_{h,r_0}$, where $\mathbb{C}[\mathfrak{t}] \to \mathbb{C}_{h,r_0}$ is given by evaluation at (h, r_0). The generalized weights of the algebra $\mathbb{C}[\mathfrak{t}]$ on $H^{\tilde{M}'}(\tilde{X}^h_N)_{h,r_0}$ are given by the differential of the above procedure, i.e., $\tilde{X}^h = \sqcup_{w\cdot h\in W_\sigma\cdot H}\tilde{X}^{h,w\cdot h}$ and $\lambda \in \mathbb{C}[\mathfrak{t}]$ acts on the component corresponding to $\tilde{X}^{h,w\cdot h}$ by the generalized weight $w \cdot h$. The proof of this observation is analogous to the the proof of the corresponding fact in K-theory, and follows from the canonical identification $H_G(X) = \mathbb{C}[\mathfrak{t}]$ and well-known facts concerning localization. The $\mathbb{C}[\mathfrak{t}]$ action on $H^{\tilde{M}'}(\tilde{X}^h_N)_{h,r_0}$ is then given by decomposing $\tilde{X}^h_N$ into connected components and determining which components lie in $\tilde{X}^{h,w\cdot h}$.

6.5 Comparison between weights for affine Hecke algebras and graded Hecke algebras. We now proceed to compare the $\mathbb{C}[\mathfrak{t}]$ and $R(T)$ actions on $H^{\tilde{M}'}_*(\tilde{X}^h_N)_{h,r_0}$ and $K^{M'}_0(\tilde{X}^s_N)_{s,v_0}$. Note that $\tilde{X}^s_N = \tilde{X}^h_N$. Let $\tilde{X}^s_N = \sqcup\tilde{X}^{s,\beta}_N$ be the decomposition into connected components. We have the following sequence of isomorphisms of vector spaces,

$$H^{\tilde{M}'}_*(\tilde{X}^{h,\beta}_N)_{h,r_0} \cong H_*(\tilde{X}^{h,\beta}_N) \cong H_*(\tilde{X}^{s,\beta}_N) \cong$$

$$\cong K_0(\tilde{X}^{s,\beta}_N) \cong K^{M'}_0(\tilde{X}^s_N)_{s,v_0}.$$

Here the first and last isomorphism are obvious since M' acts trivially on $\tilde{X}^s = \tilde{X}^h$ and the second isomorphism is obvious from the identification of the underlying topological spaces. The third isomorphism is given by the inverse of the Chern character and the fact that $H_*(\tilde{X}^{s,\beta}_N)$ vanishes in odd degrees [DLP]. It now follows from the description of the weights in the previous two sections that the dimension of the $w \cdot h$ generalized weight space of $\mathbb{C}[\mathfrak{t}]$ on $H^{M'}_*(\tilde{X}^h_N)_{h,r_0}$ is the same as the dimension of the $\sigma \cdot exp(w \cdot h)$ generalized weight space of $R(T)$ on $K^{M'}_0(\tilde{X}^s_N)_{s,v_0}$. We observed that the element $E_\sigma\theta_\lambda$ corresponds to $\theta_\lambda(\sigma)e^\lambda$ under the isomorphism of 5.7. This implies that under the isomorphism of 5.7, the $w \cdot h$ weight space of $\mathbb{C}[\mathfrak{t}]$ corresponds to the $\sigma \cdot exp(w \cdot h)$ weight space of $R(T)$. This gives the following lemma.

6.6 Lemma. *The action of $E_\sigma\theta_\lambda$ on $K^{M'}_0(\tilde{X}^s_N)_{s,v_0}$ coincides with the action of $\Phi_\sigma(E_\sigma\theta_\lambda)$ on $H^{\tilde{M}'}_*(\tilde{X}^h_N)_{h,r_0}$.*

6.7 Comparison between the two parametrizations. We observe that representations of $\mathcal{H}_\chi$ and of $H_{\sigma,\overline{\chi}}$ are classified by the same parameters. Recall the classification of irreducible representations of $H_{\sigma,\overline{\chi}}$ implicit in [Lu1]. We have the polar decomposition $s = \sigma exp(h)$ and let $r_0 = log(v_0)$. Let $K = Z_G(\sigma)$ with Lie algebra $\mathfrak{k}$ and let $\mathfrak{k}(h, r_0) = \{N \in \mathfrak{k}|[h, N] = 2r_0 \cdot N\}$. Note that $Z_K(h)$ acts on $\mathfrak{k}(h, r_0)$. For $N \in \mathfrak{k}(h, r_0)$, let $A(h, N)$ be the component group of $Z_K(h) \cap Z_K(N)$ and let $\tilde{X}$ be the flag manifold of K with fixed point set $\tilde{X}^h_N$ for the joint action of

$exp(h)$ and $exp(N)$. Then $A(h,N)$ acts on $H_*(\tilde{X}_N^h)$. Irreducible representations of $H_{\sigma,\overline{\chi}}$ are classified by $Z_K(h)$-conjugacy classes of pairs (N,ρ) such that $N \in \mathfrak{k}(h,r_0)$ and ρ is an irreducible representation of $A(h,N)$ occuring in $H_*(\tilde{X}_N^h)$.

We show that this parametrization of irreducible representations is identical to the parametrization of irreducible representations of $\mathcal{H}_\chi$ given in 5.3, where representations were parametrized by $Z_G(s)$-orbits on $\mathfrak{g}(s,v_0)$ together with certain representations of $A(s,N)$, the component group of $Z_G(s) \cap Z_G(N)$. First, we observe that $Z_G(s) = Z_K(h)$, $\mathfrak{g}(s,v_0) = \mathfrak{k}(h,r_0)$, and the component groups $A(s,N)$ and $A(h,N)$ are canonically isomorphic. The decomposition $s = \sigma \cdot exp(h)$ and the uniqueness of the polar decomposition implies that $Z_G(s) = Z_K(h)$. It is clear that $\mathfrak{k}(h,r_0) \subset \mathfrak{g}(s,v_0)$; the opposite inclusion follows easily. Indeed, if $s \cdot N = {v_0}^2 N$, then $\sigma \cdot N = N$ and $[h,N] = 2r_0 N$. The isomorphism of component groups follows from the equality $Z_G(s) \cap Z_G(N) = Z_K(h) \cap Z_K(N)$. It remains to verify that if ρ occurs in $H_*(\tilde{X}_N^h)$, then it occurs in $H_*(X_N^s)$. Recall that $\tilde{X}_N^s$ may be regarded as a union of components of X_N^s. The embedding is $Z_G(s) \cap Z_G(N)$ equivariant and $\tilde{X}_N^s = \tilde{X}_N^h$. It follows that $H_*(\tilde{X}_N^h)$ is a $A(s,N)$ submodule of $H_*(X_N^s)$. Hence, if ρ occurs in $H_*(\tilde{X}_N^h)$, it occurs in $H_*(X_N^s)$. Since there is a bijection between the set of irreducible $\mathcal{H}_\chi$ representations and irreducible $H_{\sigma,\overline{\chi}}$ representations, it follows that exactly the same representations ρ appear.

6.8 Comparison of graded Hecke algebra representations. We have the H_σ module $H_*^{\tilde{M}'}(\tilde{X}_N^h)_{h,r_0}$ Since $K_0^{M'}(X_N^t)_{s,v_0}$ is a $\mathcal{H}_\chi$ module and E_σ is a projection, it follows that $E_\sigma K_0^{M'}(X_N^t)_{s,v_0} = K_0^{M'}(\tilde{X}_N^s)_{s,v_0}$ is a $E_\sigma \mathcal{H}_\chi E_\sigma$ module. By [BM, 3.3(1), 4.3], $E_\sigma \mathcal{H}_\chi E_\sigma \cong H_{\sigma,\overline{\chi}}$. Recall also that the action of $A(s,N) = A(h,N)$ commutes with the action of $\mathcal{H}_\chi$ and in particular preserves generalized weight spaces. It follows that it commutes with E_σ. The following assertion now makes sense.

6.9 Lemma. *Assume N is in the open orbit in the $q = {v_0}^2$ eigenspace for s. Let ρ be an irreducible representation of $A(s,N)$ appearing in $H^*(\tilde{X}_N^s)$. Then*

$$K_0^{M'}(\tilde{X}_N^s)_{s,v_0}^\rho \cong H_*^{\tilde{M}'}(\tilde{X}_N^h)_{h,r_0}^\rho$$

as $H_{\sigma,\overline{\chi}}$ modules.

Proof. The hypothesis implies that $H_*^{\tilde{M}'}(\tilde{X}_N^h)_{h,r_0}^\rho$ is an irreducible H_σ module and that $K_0^{M'}(X_N^s)_{s,v_0}^\rho$ is an irreducible $\mathcal{H}_\chi$ module ([Lu1, section 8], and [KL, section 5]) It follows easily that $K_0^{M'}(\tilde{X}_N^s)_{s,v_0}^\rho$ is an irreducible $E_\sigma \mathcal{H}_\chi E_\sigma \cong H_{\sigma,\overline{\chi}}$ module. The identification of 6.6 is equivariant for the action of $A(s,N) \cong A(h,N)$, so that the two modules in the lemma have the same weights. $H_{\sigma,\overline{\chi}}$ is isomorphic to an affine Hecke algebra specialized at an appropriate central character ([Lu2]), so theorem 5.5 applies to $H_{\sigma,\overline{\chi}}$. This implies the result.

6.10 Theorem. *The Iwahori-Matsumoto shift of the classification of irreducible affine Hecke algebra modules given in [KL] is identical to the classification of irreducible graded Hecke algebra modules under the equivalence of 5.7. As a consequence, the Iwahori-Matsumoto involution corresponds to the modified Fourier transform in the Kazhdan-Lusztig classification of irreducible affine Hecke algebra representations.*

Proof. We first observe that it suffices to prove the theorem for the case of parameters of the form (N,ρ), where N is in the open orbit on $\mathfrak{g}(s,v_0)$. Indeed, we have recalled from [KL] that other irreducible affine Hecke algebra representations $\mathcal{L}(N', \rho')$ can be realized as unique irreducible quotients of irreducible representations $\hat{\mathcal{L}}(N', \rho')$ induced from affine Hecke algebras of proper Levi subalgebras where N' occurs in the open orbit on the $v_0{}^2$ eigenspace for s on the Levi subalgebra. The same result is true for graded Hecke algebras by an analogous argument. Barbasch and Moy have proved that induction commutes with the equivalence between affine Hecke algebra and graded Hecke algebra representations ([BM, 6.2 and 7.2]). It follows that it suffices to consider the case where N is in the open orbit in $\mathfrak{g}(s, v_0)$. In this case, the standard modules $K_0^{M'}(X_N^s)^\rho_{s,v_0}$ and $H_*^{\tilde{M}'}(\tilde{X}_N^h)^\rho_{h,r_0}$ are irreducible.

The equivalence between irreducible representations of $\mathcal{H}_\chi$ and $H_{\sigma,\overline{\chi}}$ is given by $V \mapsto E_\sigma V$ ([BM, remark after 3.3]). Denote the inverse equivalence by $W \mapsto J(W)$. The assertion of the theorem is that $J(H_*^{\tilde{M}'}(\tilde{X}_N^h)^\rho_{h,r_0})$ is isomorphic as a $\mathcal{H}_\chi$-module to $K_0^{M'}(X_N^s)^\rho_{s,v_0}$. This is equivalent to the assertion that $H_*^{\tilde{M}'}(\tilde{X}_N^h)^\rho_{h,r_0}$ is isomorphic to $E_\sigma K_0^{M'}(X_N^s)^\rho_{s,v_0}$ as $H_{\sigma,\overline{\chi}}$-modules. This is the assertion of 6.9.

The second assertion is now immediate since it implies modified Fourier transform corresponds to the involution $IM \circ IM \circ IM^{-1} = IM$ in the Kazhdan-Lusztig classification.

By forgetting the equivariant structure, we obtain the following corollary, which is well-known when K is a Levi factor (see [BorMac], e.g.).

6.11 Corollary. *Let $N \in \mathfrak{k}$ be nilpotent and let ρ be an irreducible representation of the component group of $A(h, N) \cong A(s, N)$. Then the W-modules $H_*(X_N)^\rho$ and $Ind_{W_\sigma}^W(H_*(\tilde{X}_N))^\rho$ are isomorphic.*

Proof. There is an embedding of W into $\mathcal{H}_\chi$ and Lusztig has shown that the induced W-module structure on $K_0^{M'}(X_N^s)^\rho_{s,v_0})$ is isomorphic to $H_*(X_N)^\rho$ ([Lu4]). The equivalence $U \mapsto J(U)$ between $H_{\sigma,\overline{\chi}}$ modules and $\mathcal{H}_\chi$-modules is given by $U \mapsto Ind_{W_\sigma}^W(U)$ on the level of W-modules [BM]. This implies the corollary.

7. Applications.

7.1. In the case of $GL(n)$, Zelevinsky has described another geometric involution on representations. In this case, every equivariant irreducible local system is a constant sheaf on an orbit, so it is enough to define an involution on orbits. Whenever a complex algebraic group M acts on a vector space V with finitely many orbits, there is a one–to–one correspondence from M–orbits on V to M–orbits on V^* [Py]. An orbit Q in V corresponds to an orbit $Z(Q)$ in V^* when the closures of their conormal bundles in $V \times V^*$ coincide. Since $\mathfrak{g}(s, v_0)^*$ can be identified with $\mathfrak{g}(s^{-1}, v_0)$, this means the correspondence $Q \mapsto Z(Q)$ induces an involution on representations which maps representations with central character (s, v_0) to those with central character (s^{-1}, v_0).

7.2. Proposition. *In the above setting, assume that for V (and hence for V^*) every M-equivariant local system is the constant sheaf on an orbit. Let $\mathcal{L}_\mathcal{O}$ be the intersection cohomology extension of the constant sheaf on $\mathcal{O}$. Then $\mathcal{F}(\mathcal{L}_\mathcal{O}) = \mathcal{L}_{Z(\mathcal{O})}$.*

Proof. We argue by induction on the dimension of the orbit. The result is clear for the $\{0\}$-orbit. Let $\mathcal{O}$ be an orbit and assume the result for all orbits contained in its closure. $\mathcal{F}(\mathcal{L}_\mathcal{O})$ is an irreducible perverse sheaf, which is determined by its support due to our assumption. Its support is $pr_{V^*}(Ch(\mathcal{F}(\mathcal{L}_{\mathcal{O}})))$, where $Ch(\mathcal{S})$ denotes the characteristic variety of a complex of sheaves $\mathcal{S}$ in $D(V^*)$. $\mathcal{L}_{\mathcal{O}}$ is conic because M contains scalar multiplication, so $Ch(\mathcal{L}_{\mathcal{O}}) = Ch(\mathcal{F}(\mathcal{L}_{\mathcal{O}}))$ [KS, 5.5.5]. $Ch(\mathcal{L}_{\mathcal{O}})$ is the union of closures of conormal bundles C_Q to some orbits $Q \subset \overline{\mathcal{O}}$. Since $\mathcal{F}(\mathcal{L}_{\mathcal{O}})$ is an irreducible perverse sheaf, its support is irreducible, but its support is $\cup_Q pr_{V^*}(C_Q)$. Thus, its support is some $pr_{V^*}(C_Q)$. Hence, $\mathcal{F}(\mathcal{L}_{\mathcal{O}}) = \mathcal{L}_{Z(Q)}$ for some orbit $Q \subset \overline{\mathcal{O}}$. If $Q \neq \mathcal{O}$, we know by induction that $\mathcal{F}(\mathcal{L}_Q) = \mathcal{L}_{Z(Q)}$. Since Fourier transform is a bijection on irreducible perverse sheaves, $\mathcal{L}_{\mathcal{O}} = \mathcal{L}_Q$, which is a contradiction. □

7.3. Corollary. *The orbit correspondence described by Zelevinsky for $GL(n)$ induces the same involution on representations as the Iwahori–Matsumoto involution.*

Proof. This follows from 6.10 and the above proposition. □

7.4. Remark. The above corollary was proved by Moeglin and Waldspurger by a very different argument. Their result applies to arbitrary smooth, irreducible representations.

7.5. Remark. It is quite easy to see that Fourier transform need not coincide with the orbit correspondence in cases where there are non–trivial local systems. This happens when $G = Sp(2)$, $P = B$, and $s = (v_0, v_0, {v_0}^{-1}, {v_0}^{-1})$ and $v = v_0$. Indeed, in this case $Z(s) = GL(2)$, V is symmetric 2×2 matrices, and the action is the adjoint action. There are three orbits, $\mathcal{O}_0$, $\mathcal{O}_1$, and $\mathcal{O}_2$ corresponding to matrices of rank zero, one and two. $\mathcal{O}_1$ and $\mathcal{O}_2$ each carry nontrivial local systems. Using the obvious identification between orbits on V and on V^*, the orbit correspondence interchanges $\mathcal{O}_0$ and $\mathcal{O}_2$ and preserves $\mathcal{O}_1$. However, $\mathcal{F}(\mathcal{L}_{\mathcal{O}_1})$ is the nontrivial local system on the open orbit $\mathcal{O}_2$. Indeed,

$$Ch(\mathcal{L}_{\mathcal{O}_1}) = \overline{T^*{\mathcal{O}_1}} + \overline{T^*{\mathcal{O}_0}},$$

so

$$Ch(\mathcal{F}(\mathcal{L}_{\mathcal{O}_1})) = \overline{T^*{\mathcal{O}_1}} + \overline{T^*{\mathcal{O}_2}},$$

which implies $\mathcal{F}(\mathcal{L}_{\mathcal{O}_1})$ is as stated above.

Note that in the above example, the characteristic cycle of a tempered representations (corresponding to the perverse extension of a local system living on the open orbit) does not contain the conormal bundle to the orbit through $\{0\}$. This is a general phenomenon.

7.6 Proposition. *Let $V \cong \mathcal{L}(s, N, \rho)$ be a tempered representation, and assume $\mathfrak{g}(s, v_0)$ is not zero. Let $IC(s, N, \rho)$ denote the perverse extension of the local system on $\mathfrak{g}(s, v_0)$ corresponding to the parameters (N, ρ). Then $Ch(IC(s, N, \rho))$ does not contain the conormal bundle to the orbit through $\{0\}$.*

Proof. Assume $Ch(IC(s, N, \rho))$ contains the conormal bundle to $\{0\}$. Since $IC(s, N, \rho)$ and $\mathcal{F}(IC(s, N, \rho))$ have the same characteristic variety, it follows that the support of $\mathcal{F}(IC(s, N, \rho))$ contains $\mathfrak{g}(s, v_0)^* = pr_{\mathfrak{g}(s,v_0)^*}(T_0^*(\mathfrak{g}(s, v_0)))$. Thus, $\mathcal{F}(IC(s, N, \rho))$ is the perverse extension of a local system on the open orbit, so it corresponds to a tempered representation [KL, 8.2]. By our main theorem, it follows that V and $IM(V)$ are both tempered, so by our remarks in 5.3, $\mathfrak{g}(s, v_0) = \{0\}$. This is a contradiction.

7.7 Remark. In the language of [Vo], this means that the Arthur packet of the representation $\mathcal{L}(s, 0, triv)$ (the Langlands quotient of the full principal series) does not contain a tempered representation. Clearly, one can obtain considerably more precise results in this direction.

Appendix A. Convolution algebras (after Ginzburg)

A.1 The algebra $E_{\mathcal{S}}$. Let E be a space and let $\mathcal{S} \in D(E)$ be a semisimple complex of sheaves, so

$$\mathcal{S} = \oplus_{\pi \in \hat{\mathcal{S}}} V_\pi \otimes \pi$$

for some set $\hat{\mathcal{S}}$ of irreducible perverse sheaves and some finite dimensional graded vector spaces V_π. Then the finite dimensional graded algebra

$$E_{\mathcal{S}} \stackrel{\text{def}}{=} Ext^*_{D(E)}(\mathcal{S}, \mathcal{S}) = \oplus_{\pi,\pi\prime} Hom^*_{\mathbb{C}}(V_\pi, V_{\pi\prime}) \otimes Ext^*_{D(E)}(\pi, \pi\prime)$$

has another useful grading $E_{\mathcal{S}}{}^p \stackrel{\text{def}}{=} \oplus Hom_{\mathbb{C}}(V_\pi, V_\pi\prime) \otimes Ext^p(\pi, \pi\prime)$. By [BBD], $E_{\mathcal{S}}{}^p = 0$ for $p < 0$. But then since $E_{\mathcal{S}}$ is finite dimensional, the irreducible modules for $E_{\mathcal{S}}$ and for $E_{\mathcal{S}}{}^0 = \oplus_\pi End_{\mathbb{C}}(V_\pi)$ are the same. This gives the following lemma.

A.2 Lemma [Gi]. *The set of irreducible $E_{\mathcal{S}}$ modules, $\{V_\pi, \pi \in \hat{\mathcal{S}}\}$ is in one to one correspondence with the set $\hat{\mathcal{S}}$ of irreducible constituents of $\mathcal{S}$.*

A.3. Let $\pi : V \to E$ be a proper map and let $\mathcal{L} \in D(V)$ be a semisimple complex of geometric origin. The decomposition theorem guarantees that $\pi_*(\mathcal{L})$ is also semisimple. Hence, irreducible modules for its Ext-algebra are indexed by its irreducible constituents.

A.4 Lemma. *Let $\Lambda = V \underset{E}{\times} V \overset{p,q}{\to} V$ be the projections. Then*

$$Ext^*_{D(E)}(\pi_*\mathcal{L}, \pi_*\mathcal{L}) = Ext^*_{D(\Lambda)}(p^*\mathcal{L}, q^!\mathcal{L}).$$

Proof.

$$\begin{aligned} Ext^*_{D(E)}(\pi_*\mathcal{L}, \pi_*\mathcal{L}) &= Ext^*_{D(V)}(\pi^*\pi_!\mathcal{L}, \mathcal{L}) \\ = Ext^*_{D(V)}(q_!p^*\mathcal{L}, \mathcal{L}) &= Ext^*_{D(\Lambda)}(p^*\mathcal{L}, q^!\mathcal{L}) \end{aligned}$$

by the properness of π, adjunction, and base change.

A.5 Remark. These constructions can be made in the equivariant derived category. Lemma A.2 fails since the algebra $E_{\mathcal{S}}$ can be infinite dimensional but the remaining statements are valid in the equivariant derived category by the same arguments.

Appendix B. Localization for equivariant complexes

B.1. In this appendix, we give the easy generalization B.2 of the localization theorem to objects in the equivariant derived category (following [Gi], [Se], [Jo], [Ki] and others) and its slight variation B.7. Let T be a torus and let X be a T-variety which admits a locally closed T-embedding into a smooth projective T-variety $\tilde{X}$. Let $\mathcal{A} \in D_T(X)$ and let $N(\mathcal{A}, \tilde{X})$ be the set of characters of T which appear in the normal bundles of the connected components C of the fixed point set $\tilde{X}^T$ which meet the support of $\mathcal{A}$. For a subtorus S of T, let $N_S(\mathcal{A},\tilde{X}) = \{\chi \in N(\mathcal{A},\tilde{X}) | \chi|S \neq 1\}$ and let $\rho : X^S \hookrightarrow X$ be the inclusion of the fixed point set. For α in $\mathfrak{t} = Lie(T)$, we write $H^*_\alpha(X,\mathcal{A}) \stackrel{\text{def}}{=} H^*_T(X,\mathcal{A}) \otimes_{H^*_T} \mathbb{C}_\alpha$, where $\mathbb{C}_\alpha = H^*_T/m_\alpha$, with m_α the maximal ideal defined by evaluation at α. Via the embedding of the character group $X^*(T) \subset \mathfrak{t}^*$, we regard $N_S(\mathcal{A},\tilde{X})$ as a subset of $\mathfrak{t}^*$.

B.2 Theorem.

(a) Let $\mathcal{A} \in D_T(X)$ and let $\tilde{H}_T$ be localization of the multiplicative set in H^*_T generated by $N_S(\mathcal{A},\tilde{X})$. Then the canonical maps

$$H^*_T(X,\rho_!\rho^!\mathcal{A}) \to H^*_T(X,\mathcal{A}), \quad H^*_{c,T}(X,\rho_*\rho^*\mathcal{A}) \leftarrow H^*_{c,T}(X,\mathcal{A})$$

become isomorphisms after one applies $-\otimes_{H^*_T} \tilde{H}_T$.

(b) If $\alpha \in \mathfrak{s} = Lie(S) \subset \mathfrak{t}$ is sufficiently generic (i.e., $<\chi,\alpha> \neq 0$, $\chi \in N_S(\mathcal{A},\tilde{X})$), then $H_\alpha(X,\rho_!\rho^!\mathcal{A}) \cong H_\alpha(X,\mathcal{A})$ and $H_{c,\alpha}(X,\rho_*\rho^*\mathcal{A}) \cong H_{c,\alpha}(X,\mathcal{A})$.

B.3 Remark. One gets the usual localization theorem in equivariant cohomology by letting $\mathcal{A}$ be the constant sheaf and by inverting all characters which are nontrivial on S.

Part (b) follows from part (a) since the specialization map $H^*_T \to \mathbb{C}_\alpha$ factors through $\tilde{H}_T$. Moreover, we only need to prove the first statement in (a) since the second statement follows from the proof by using Verdier duality. We will say that the localization principle holds for the pair $(X,\mathcal{A})$ if $H^*_T(X,\rho_!\rho^!\mathcal{A}) \to H^*_T(X,\mathcal{A})$ becomes an isomorphism after tensoring with the above $\tilde{H}_T$.

B.4 Lemma [Mayer-Vietoris reduction]. *Let $Y \stackrel{i}{\hookrightarrow} X$ be T-equivariant and closed. If the localization principle holds for two of the pairs $(X,\mathcal{A})$, $(Y,i^!\mathcal{A})$, and $(X-Y,\mathcal{A}|X-Y)$ then it holds for the third.*

Proof. Let $j : U = X - Y \hookrightarrow X$ be the inclusion and consider the distinguished triangle $i_!i^!\mathcal{A} \to \mathcal{A} \to j_*j^*\mathcal{A}$. Applying $\rho_!\rho^!$ to this triangle yields the distinguished triangle $i_!\rho_{Y!}\rho_Y{}^!i^!\mathcal{A} \to \rho_!\rho^!\mathcal{A} \to j_*\rho_{U!}\rho_U{}^!j^*\mathcal{A}$ and the adjunction morphism $\rho_!\rho^! \to id_{D_T(X)}$ induces a map of the second triangle to the first. Take equivariant cohomology and tensor with $\tilde{H}_T$ to get a map of exact triangles of $\tilde{H}_T$-modules. Our lemma now follows from the five lemma.

Proof of B.2. Factor the embedding $X \stackrel{\mu}{\hookrightarrow} \tilde{X}$ as $X \hookrightarrow \overline{X} \hookrightarrow \tilde{X}$ and apply Mayer-Vietoris reduction twice to replace $(X,\mathcal{A})$ with $(\tilde{X},\mu_*\mathcal{A})$. The Bialynicki-Birula decomposition of $\tilde{X}$ with respect to T realizes $\tilde{X}$ as a finite union of locally closed strata X_i, where each X_i is a T-equivariant vector bundle over ${X_i}^T$ [B-B]. By Mayer-Vietoris reduction, it suffices to prove the localization principle for each X_i, so that we may assume X is a T-equivariant vector bundle over X^T. Trivialize X over an open cover of X^T by open sets U_i, and then use Mayer-Vietoris reduction for X^T, U_i, and $X^T - U_i$ and induction to reduce to the case where X is trivial. Thus, we may assume $X = X^T \times \prod \mathbb{C}_\chi$, where the product is over characters appearing in the normal bundle to the components of X^T. Observe that X^S is a product of X^T and the $\mathbb{C}_\chi$'s with $\chi|S = 1$. The theorem now follows from the following lemma.

B.5 Lemma. *Let Y be a T-variety, χ a character of T and $i : Y \hookrightarrow Y \times \mathbb{C}_\chi$ the canonical embedding. Then for any $\mathcal{A} \in D_T(Y \times \mathbb{C}_\chi)$, the map $H_T^*(Y, i^!\mathcal{A}) \to H_T^*(Y \times \mathbb{C}_\chi, \mathcal{A})$ becomes an isomorphism after one inverts χ.*

Proof. Let $Z = Y \times \mathbb{C}_\chi$. The Mayer-Vietoris exact sequence shows that it is enough to prove that $H_T^*(Z - Y, \mathcal{A})$ is killed by localization. For $B = ker(\chi)$, we have $Z - Y = T \times_B (Y \times 1)$, hence $H_T^*(Z - Y, \mathcal{A}|Z - Y) = H_B^*(Y, \mathcal{A})$ and this last group is zero when one inverts χ.

B.6 . Let G be a connected reductive algebraic group acting on a variety X. For any $\mathcal{A} \in D_G(X)$ we call $\Pi_G^*(X, \mathcal{A}) = H_G^*(X, \mathcal{A}) \otimes_{H_G^*} \mathbb{C}_0$ the G-primitive part of the cohomology of $\mathcal{A}$. This is precisely $E_\infty^{0,*}$, the limit of the left column of the spectral sequence $E_2^{p,q} = H^q(X, \mathcal{A}) \otimes H_G^p \to E_\infty^{p,q} = Gr_p H_G^{p+q}(X, \mathcal{A})$, hence it is a graded subobject of $H^*(X, \mathcal{A})$. G-primitive homology of a local system is defined by the analogous construction.

B.7 Theorem. *Let $\alpha \in \mathfrak{g}$ be semisimple, S the torus it generates in G and $X^S \overset{\rho}{\hookrightarrow} X$. Let L be a Levi subgroup of $Z_G(\alpha)$.*

(a) For $\mathcal{A} \in D_G(X)$ one has $H_G^*(X, \mathcal{A}) \otimes_{H_G^*} \mathbb{C}_\alpha \cong H_L^*(X, \rho_!\rho^!\mathcal{A}) \otimes_{H_L^*} \mathbb{C}_\alpha$.

(b) If the L-structure on $\rho^!\mathcal{A}$ is S-trivial, i.e., it comes from a L/S-structure,

$$H_G^*(X, \mathcal{A}) \otimes_{H_G^*} \mathbb{C}_\alpha \cong \Pi_{L/S}^*(X^S, \rho^!\mathcal{A}) \subset H^*(X^S, \rho^!\mathcal{A}).$$

(c) For any G-equivariant local system $\mathcal{L}$ on X there is a canonical isomorphism

$$H_*^G(X, \mathcal{L}) \otimes_{H_G^*} \mathbb{C}_\alpha \cong \Pi_*^{L/S}(X^S, \mathcal{L}).$$

Proof. The isomorphism $H_T^*(X, \mathcal{A}) \cong H_*^G(X, \mathcal{A}) \otimes_{H_G^*} H_T^*$ implies that $H_G^*(X, \mathcal{A}) \otimes_{H_G^*} \mathbb{C}_\alpha \cong H_*^T(X, \mathcal{A}) \otimes_{H_T^*} \mathbb{C}_\alpha$. Similarly, we obtain $H_L^*(X, \rho_!\rho^!\mathcal{A}) \otimes_{H_L^*} \mathbb{C}_\alpha \cong H_*^T(X, \rho_!\rho^!\mathcal{A}) \otimes_{H_T^*} \mathbb{C}_\alpha$. Thus, (a) follows from B.2. To prove (b), note that the assumption of S-triviality implies $H_*^L(X^S, \rho^!\mathcal{A}) = H_*^{L/S}(X^S, \rho^!\mathcal{A}) \otimes_{H_{L/S}^*} H_L^*$. Then (b) follows from (a) and the fact that the image of α in $\mathfrak{l}/\mathfrak{s}$ is zero. Part (c) follows from (b) due to the S-triviality of L-structures on $\mathcal{L}|X^S$ and the dualizing sheaf ω_{X^S}.

B.8 The compact Kaehler case. Assume more generally that X admits a locally closed T-embedding into a compact Kaehler manifold $\tilde{X}$. We also assume that $\tilde{X}$ has T-fixed points, or equivalently the compact torus action on $\tilde{X}$ is Hamiltonian (the equivalence follows from [Fr]). The results and proofs above are valid with a slight modification since $\tilde{X}$ has a Bialynicki-Birula type decomposition [CS].

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