NOTES ON LINEAR GROUPS AND THEIR ACTIONS ON FLAG SPACES AND GRASSMANNIANS

Contents

1. Introduction

These notes discuss some subgroups of $GL(n, F)$ (for a field F), and actions of these groups on certain sets, including Grassmannians and flag spaces. They were prepared for Basic Algebra, 60210, discussion on group actions from September 2018.

2. Linear algebra remarks

Let F be a field, and let $V = F^n$ be the space of column vectors with standard basis $e_1, \ldots, e_n$. We sometimes regard column vectors as n by 1 matrices. Let $A = (a_{ij})$ be a matrix in $M(n, F)$ with entry a_{ij} in row i and column j. For $A \in M(n, F)$, we let $A(v) \in F^n$ equal $A \times v$, where $A \times v$ denotes the product of a n by n matrix with a n by 1 vector (i.e., a column vector), which is a n by 1 vector, and hence a column vector.

Remark 2.1. *For $A = (a_{ij}) \in M(n, F)$, then*

$$A(e_k) = \sum_{r=1}^{n} a_{rk} e_r.$$

This is an easy computation.

Notation 2.2. *Let $A \in M(n, F)$ and let $U \subset V$ be a subspace, i.e., a linear subspace. We let*

$$A \cdot U := \{A(v) : v \in U\}.$$

Since A gives a linear map from V to V, it follows easily that $A \cdot U$ is a subspace of V.

Remark 2.3. *If $A, B \in M(n, F)$ and $U \subset V$ is a subspace, then $(A \times B) \cdot U = A \cdot (B \cdot U)$. Indeed, if $v \in U$, then*

$$(A \times B) \cdot U := \{(A \times B)(v) : v \in U\} = A \cdot \{B(v) : v \in U\} = A \cdot (B \cdot U).$$

Definition 2.4. *Fix k with $1 \leq k \leq n$. Let $\mathrm{Gr}(k, V)$ be the set of all k-dimensional subspaces U of V.*

In a geometry or topology class, you may have seen a discussion of $\mathrm{Gr}(k, V)$, where $\mathrm{Gr}(k, V)$ is defined as above and given the structure of a topological space. For this algebra class, we just regard $\mathrm{Gr}(k, V)$ as a set.

Remark 2.5. *If $g \in GL(n, F)$ and $U \in \mathrm{Gr}(k, V)$, then $g \cdot U$ is also a k-dimensional subspace. Indeed, if $\{v_1, \ldots, v_k\}$ is a basis of U, then $\{g(v_1), \ldots, g(v_k)\}$ is a linearly independent subset of V, as follows easily from the fact that g is invertible. From the fact that $\{v_1, \ldots, v_n\}$ spans U, it follows easily that $\{g(v_1), \ldots, g(v_k)\}$ spans $g \cdot U$. Hence, $\{g(v_1), \ldots, g(v_k)\}$ spans $g \cdot U$ is a basis of $g \cdot U$, so that $g \cdot U \in \mathrm{Gr}(k, V)$.*

Remark 2.6. *$GL(n, F)$ acts on $\mathrm{Gr}(k, V)$ via $(g, U) \mapsto g \cdot U$, where $g \in GL(n, F)$ and $U \in \mathrm{Gr}(k, V)$. Note that $g \cdot U \in \mathrm{Gr}(k, V)$ by Remark 2.5. We verify that this is a group action. Indeed, if $g, h \in GL(n, F)$ and $U \in \mathrm{Gr}(k, V)$, then by Remark 2.3, $(gh) \cdot U = g \cdot (h \cdot U)$. For $e = \mathrm{Id}_n$, the n by n identity, $e \cdot U = \{\mathrm{Id}_n(v) : v \in U\} = \{v : v \in U\} = U$. Hence, the $GL(n, F)$ action is a group action.*

Exercise 2.7. *(i) The action of $GL(n, F)$ on $\mathrm{Gr}(k, V)$ is transitive.*

(ii) Let $F_k = Fe_1 + \cdots + Fe_k$ be the span of the first k standard vectors $e_1, \ldots, e_k$. Then the stabilizer of the $GL(n, F)$ action of the point F_k is the subgroup

$$\{A = (a_{ij}) \in GL(n, F) : a_{ij} = 0 \ \forall \ i = k + 1, \ldots, n, \ j = 1, \ldots, k\}.$$

Proposition 2.8. *Let F be a field with a finite number q of elements. Then*

$$|GL(n, F) = \prod_{i=0,\ldots,n-1} (q^n - q^i).$$

We prove this in class using the action of $GL(n, F)$ on F^n, showing that if $X = F^n - \{0\}$, then the $GL(n, F)$ action on X is transitive, and by computing the stabilizer of the vector e_1.

We now introduce three subsets of the set of matrices.

Definition 2.9. *Let*

(i) $\mathfrak{b}(n, F) = \{A \in M(n, F) : a_{ij} = 0 \forall \ i > j\}$,

(ii) $\mathfrak{n}(n, F) = \{A \in M(n, F) : a_{ij} = 0 \forall \ i \geq j\}$,

(iii) $\mathfrak{d}(n, F) = \{A \in M(n, F) : a_{ij} = 0 \forall \ i \neq j\}$.

Note that $\mathfrak{n}(n, F)$ and $\mathfrak{d}(n, F)$ are subsets of $\mathfrak{b}(n, F)$.

Remark 2.10. *If $A \in \mathfrak{b}(n, F)$, then we can write $A = A_d + A_u$ uniquely, where $A_d \in \mathfrak{d}(n, F)$ and $A_u \in \mathfrak{n}(n, F)$, just by taking the diagonal and strictly upper triangular parts of the upper triangular matrix A.*

As above, for $i = 1, \ldots, n$, let F_i be the span of the first i standard vectors $e_1, \ldots, e_i$. In particular, $F_i = \{\sum_{j=1,\ldots,n} a_j e_j : a_j \in F, a_j = 0 \forall\ j > i.\}$. We set $F_j = 0$ for $j \leq 0$.

Lemma 2.11. *Let $A \in M(n, F)$. Then*

(i) $\mathfrak{b}(n, F) = \{A \in M(n, F) : A \cdot F_i \subset F_i, i = 1, \ldots, n.\}$

(ii) $\mathfrak{n}(n, F) = \{A \in M(n, F) : A \cdot F_i \subset F_{i-1}, i = 1, \ldots, n.\}$

Proof. We prove (i). First we claim that $A \cdot F_i \subset F_i$ for all $i = 1, \ldots, n$ if and only if $A(e_i) \in F_i$ for $i = 1, \ldots, n$. Indeed, one direction is clear, and now suppose that $A(e_i) \in F_i$ for $i = 1, \ldots, n$. Since $F_k \subset F_i$ for $k < i$, it follows that if $k \leq i$, $A(e_k) \in F_i$. Since $e_1, \ldots, e_i$ spans F_i, it follows that $A(v) \in F_i$ for all $v \in F_i$, and this implies $A \cdot F_i \subset F_i$, which establishes the claim.

Since by Remark 2.1, $A(e_i) = \sum_{r=1,\ldots,n} a_{ri} e_r$, we see that $A(e_i) \in F_i$ if and only if $\sum_{r=1,\ldots,n} a_{ri} e_r \in F_i$ if and only if $a_{ri} = 0$ for $r = i+1, \ldots, n$. Hence, $A \in \mathfrak{b}(n, F)$ if and only if $A(F_i) \subset F_i$ for $i = 1, \ldots, n$, and this proves (i).

The proof of (ii) proceeds in essentially the same manner as the proof of (i), and we leave the details to the reader.

Q.E.D.

Lemma 2.12. *(i) If $A, X \in \mathfrak{b}(n, F)$, then $A \times X \in \mathfrak{b}(n, F)$*

(ii) If $A \in \mathfrak{b}(n, F)$ and $X \in \mathfrak{n}(n, F)$, then $A \times X$ and $X \times A \in \mathfrak{n}(n, F)$.

(iii) If $A, X \in \mathfrak{d}(n, F)$, then $A \times X \in \mathfrak{d}(n, F)$.

Proof. By Lemma 2.11(i), $A \cdot F_i \subset F_i$ for all i, and $X \cdot F_i \subset F_i$ for all i. Thus, $(A \times X) \cdot F_i \subset F_i$ for all i, so $A \cdot X \in \mathfrak{b}(n, F)$, again by Lemma 2.11(i), and this proves (i). For (ii), let $A \in \mathfrak{b}(n, F)$ and let $X \in \mathfrak{n}(n, F)$. By Lemma 2.11, $A \cdot F_i \subset F_i$ and $X \cdot F_i \subset F_{i-1}$ for $i = 1, \ldots, n$. Thus, for $i = 1, \ldots, n$, by Remark 2.3,

$$(A \times X) \cdot (F_i) = A \cdot (X \cdot (F_i)) \subset A \cdot F_{i-1} \subset F_{i-1},$$

so that $A \times X \in \mathfrak{n}(n, F)$ by Lemma 2.11(ii). The proof that $X \times A \in \mathfrak{n}(n, F)$ is similar, and this proves (ii). Assertion (iii) is easily proved by direction computation.

Q.E.D.

3. Subgroups of $GL(n, F)$

We introduce 3 subsets of $GL(n, F)$ as follows.

Definition 3.1. *(i) Let $B(n, F) := \mathfrak{b}(n, F) \cap GL(n, F)$.*

(ii) Let $N(n, F) := \{\mathrm{Id}_n + X : X \in \mathfrak{n}(n, F)\}$.

(iii) Let $D(n, F) := \mathfrak{d}(n, F) \cap GL(n, F)$.

Definition 3.2. *Let*

$$\mathrm{Fl}(n) := \{(V_1, V_2, \ldots, V_n) : V_i \in \mathrm{Gr}(i, V) \forall\ i = 1, \ldots, n,\ V_i \subset V_{i+1} \forall i = 1, \ldots, n-1\}.$$

We call $\mathrm{Fl}(n)$ *the flag space of* F^n*.* $\mathrm{Fl}(n)$ *is the collection of all nested sequences of subspaces* $V_1 \subset V_2 \subset \cdots \subset V_n$*.*

Recall the subspaces F_i of F^n, where as before F_i is the span of the vectors $e_1, \ldots, e_i$. Note that $\dim(F_i) = i$ and $F_i \subset F_{i+1}$ for $i = 1, \ldots, n-1$, so if $\mathcal{F}_0 = (F_1, F_2, \ldots, F_n)$, then $\mathcal{F}_0 \in \mathrm{Fl}(n)$.

Note that $G = GL(n, F)$ acts on $\mathrm{Fl}(n)$ by the formula $g \cdot (V_1, \ldots, V_n) = (g \cdot V_1, \ldots, g \cdot V_n)$ for $g \in G$ and $(V_1, \ldots, V_n) \in \mathrm{Fl}(n)$. By Remark 2.5, $g \cdot V_i \in \mathrm{Gr}(i, F^n)$, and since $V_i \subset V_{i+1}$ implies $g \cdot V_i \subset g \cdot V_{i+1}$, it follows that $g \cdot (V_1, \ldots, V_n) \in \mathrm{Fl}(n)$. By the assertions of Remark 2.6, it follows that the map $(g, (V_1, \ldots, V_n)) \mapsto (g \cdot V_1, \ldots, g \cdot V_n)$ is a G-action on $\mathrm{Fl}(n)$.

Proposition 3.3. *(i)* $B(n, F)$ *is the stabilizer subgroup of* $\mathcal{F}_0$*.*

(ii) $N(n, F)$ *is a normal subgroup of* $B(n, F)$*, and* $D(n, F)$ *is a subgroup of* $B(n, F)$*.*

(iii) If $|F| = q$ *is finite, then*

$$|B(n, F)| = (q-1)^n \cdot q^{\frac{n(n-1)}{2}}.$$

$$|N(n, F)| = q^{\frac{n(n-1)}{2}}.$$

$$|D(n, F)| = (q-1)^n.$$

Proof. For (i), let $g \in GL(n, F)$. Then $g \cdot \mathcal{F}_0 = \mathcal{F}_0$ if and only if $g \cdot F_i = F_i \forall\ i = 1, \ldots, n$. Note that $g \cdot F_i \subset F_i$ if and only if $g \cdot F_i = F_i$. Indeed, one implication is trivial, and for the other implication, recall that by Remark 2.5, $\dim(g \cdot F_i) = i$. Hence, if $g \cdot F_i \subset F_i$, then $g \cdot F_i = F_i$ since for a i-dimensional subspace U of a i-dimensional subspace W, $U = W$. It follows that $g \cdot \mathcal{F}_0 = \mathcal{F}_0$ if and only if $g \cdot F_i \subset F_i$ for all i, and this is equivalent to requiring that $g \in \mathfrak{b}(n, F)$ by Lemma 2.11 (i). Hence, g stabilizes $\mathcal{F}_0$ if and only if $g \in GL(n, F) \cap \mathfrak{b}(n, F) = B(n, F)$. This proves assertion (i).

For assertion (ii), let $A \in B(n, F)$. Then $A = A_d + A_u$ with $A_d \in \mathfrak{d}(n, F)$ and $A_u \in \mathfrak{n}(n, F)$ by Remark 2.10. Note that $\det(A) = \det(A_d)$ by standard facts about determinants. Since $A \in GL(n, F)$, $\det(A) \in F^\times$, so $A_d \in GL(n, F)$ and $A_d \in D(n, F)$. We define $\phi : B(n, F) \to D(n, F)$ by $\phi(A) = A_d$. We will prove (ii) by showing that ϕ is a group homomorphism with kernel $N(n, F)$ and image $D(n, F)$. Let $A = A_d + A_u$ as above, and similarly, for $C \in B(n, F)$, then $C = C_d + C_u$ with $C_d \in D(n, F)$ and $C_u \in \mathfrak{n}(n, F)$. Thus, $\phi(A) = A_d$ and $\phi(C) = C_d$. Then

$$A \times C = (A_d + A_u) \times (C_d + C_u) = A_d \times C_d + A_d \times C_u + A_u \times C_d + A_u \times C_u.$$

By Lemma 2.12, $A \times C \in \mathfrak{b}(n, F)$, $A_d \times C_d \in \mathfrak{d}(n, F)$, and $A_d \times C_u + A_u \times C_d + A_u \times C_u \in \mathfrak{n}(n, F)$. By uniqueness of the decomposition in Remark 2.10, we see that $(A \times C)_d = A_d \times C_d$. Since $A_d, C_d \in D(n, F)$, we see that $A_d \times C_d \in D(n, F)$, so that $\phi(A \times C) = A_d \times C_d$, and thus ϕ is a group homomorphism. The image of ϕ is evidentally in $D(n, F)$, and if

$A \in D(n,F)$, then $A \in B(n,F)$, and $\phi(A) = A$. Thus, the image of ϕ is $D(n,F)$, and $D(n,F)$ is a subgroup. Finally, the kernel of ϕ is the set

$$\{A \in B(n,F) : \phi(A) = \mathrm{Id}_n\} = \{A \in B(n,F) : A_d = \mathrm{Id}_n\} = \{\mathrm{Id}_n + X : X \in \mathfrak{n}(n,F)\} = N(n,F).$$

Hence, $N(n,F)$ is normal because it is the kernel of a group homomorphism.

To prove (iii), note that $D(n,F) \cap N(n,F) = \{\mathrm{Id}_n\}$. Since $N(n,F)$ is normal in $B(n,F)$ by (ii), we obtain a group homomorphism $\sigma : D(n,F) \to \mathrm{Aut}(N(n,F))$ by $\sigma(g)(x) = gxg^{-1}$ with $g \in D(n,F)$ and $x \in N(n,F)$, by Problem 6(i) from Problem Set 4. By Problem 6(ii) of Problem Set 4, the map

$$\chi : N(n,F) \ltimes_\sigma D(n,F) \to B(n,F),\ \chi(x,g) = x \cdot g,$$

is an injective group homomorpism with image $N(n,F)D(n,F)$. We show that $B(n,F) = N(n,F)D(n,F)$. Indeed, if $A \in B(n,F)$, then as above, $A = A_d + A_u = A_d \times (\mathrm{Id}_n + A_d^{-1} \times A_u)$ and $A_d^{-1} \times A_u \in \mathfrak{n}(n,F)$ by Lemma 2.12 (ii). Thus, $y = (\mathrm{Id}_n + A_d^{-1} \times A_u) \in N(n,F)$. Hence,

$$A = A_d \times y = (A_d \times y \times A_d^{-1}) \times A_d \in N(n,F)D(n,F),$$

since $N(n,F)$ is normal in $B(n,F)$. Thus,

$$\chi : N(n,F) \ltimes_\sigma D(n,F) \to B(n,F) \tag{3.1}$$

is a group isomorphism. Since $N(n,F) \ltimes_\sigma D(n,F) = N(n,F) \times D(n,F)$ as sets, we conclude that

$$|B(n,F)| = |D(n,F)| \cdot |N(n,F)|.$$

Note that $D(n,F) \cong (F^\times)^n$, so $|D(n,F)| = (q-1)^n$, as $|F^\times| = q-1$. Also, the map $\mathfrak{n}(n,F) \to N(n,F)$, $X \mapsto \mathrm{Id}_n + X$, is clearly bijective, so $|N(n,F)| = |\mathfrak{n}(n,F)|$. But $\mathfrak{n}(n,F)$ is a F-vector space with basis $\{E_{ij} : i = 1, \ldots, n-1;\ j = i+1, \ldots, n\}$ so that $\dim(\mathfrak{n}(n,F)) = n-1+n-2+\cdots+2+1 = \frac{n(n-1)}{2}$, and thus $|\mathfrak{n}(n,F)| = |F^{(n(n-1))/2}| = q^{n(n-1)/2}$. Assertion (iii) follows.

Q.E.D.