Due Wednesday, September 14

INSTRUCTIONS: Do 9 of these 13 problems.

1. Let $p$ be a prime of the form $3k + 1$ with $k \in \mathbb{Z}_{>0}$. Prove that $p$ is of the form $6k + 1$ with $k \in \mathbb{Z}_{>0}$.
2. Problem 17 of 1.3, p. 29.
3. For $a, b \in \mathbb{Z}$, not both 0, prove that $(a^2, b^2) = ((a, b))^2$.
4. Prove that there are infinitely many primes of the form $4n + 3$, and that there are infinitely many primes of the form $6n + 5$.
5. Prove that $n$ divides $(n−1)!$ for every composite number $n > 4$. If $p$ is a prime, does $p$ divide $(p−1)!$?
6. Show that $n^4 + 4$ is a composite number for every $n > 1$.
7. Problem 43, of 1.3, p. 33.
8. Let $a = 38808$ and let $b = 1887600$.
   (a) Use the Euclidean algorithm to compute $(a, b)$.
   (b) Find the prime factorizations of $a$ and $b$.
   (c) Use your answer in part (b) to find the prime factorizations of $(a, b)$ and $[a, b]$.
9. List all numbers from 1 to 150 that are congruent to 5 mod 23.
10. For each of the following integers $n$, find a reduced residue system modulo $n$:
    (a) $n = 7$
    (b) $n = 14$
    (c) $n = 20$.
11. Let $n$ be an integer. Prove the following assertions:
    (a) $n^{12} − 1$ is divisible by 7
    (b) $n^{16} − 1$ is divisible by 17
    (c) $n^{80} − 1$ is divisible by 17.
12. Let $a_n a_{n−1} \ldots a_1 a_0$ be the decimal representation of a number $m$ with $n + 1$ digits. Prove that 11 divides $m$ if and only if 11 divides

$$\sum_{i=0}^{n} (-1)^i a_i.$$

13. For a number $m$ with 4 digits in the form $a_3 a_2 a_1 a_0$, give a criterion for when 7 divides $m$. How much time does it take to apply compared to just punching the numbers into a calculator? (hint: for a number $m$ with 2 digits, $m = a_1 a_0$, then $m$ is divisible by 7 if and only if 7 divides $a_0 + 3a_1$).
Extra Credit: Prove that if $y$ is a real number, and $\mathcal{N}$ be the collection of positive integers $k$ such that if $p$ is a prime factor of $k$, then $p \leq y$. Prove that

$$\prod_{p \leq y} (1 + p^{-1} + p^{-2} + \ldots) = \sum_{k \in \mathcal{N}} \frac{1}{k}.$$

Note that both sides involve infinite sums, so you should check convergence.