HOMEWORK 1, MATH 60210, BASIC ALGEBRA, DUE WEDNESDAY, AUGUST 29
INSTRUCTOR, SAM EVENS, FALL 2018

INSTRUCTIONS: Do 7 of these 11 problems. Problems with a star are more challenging.

1. Ash, 1.1, problem 8, i.e., let G be a group and let $a, b \in G$ and assume $ab = ba$. Let $|a| = m$ and $|b| = n$. If m and n are relatively prime, show that $|ab| = mn$ and $\langle a \rangle \cap \langle b \rangle = \{e\}$.

2*. Ash, 1.1, problem 9, i.e., show that if G is a finite abelian group, then there is an element $a \in G$ such that the order of a is the least common multiple of the orders of elements of G (hint: first show that if a and b are as in problem 1, then ab has order $[m, n]$ in the language of problem 8).

3. Ash, 1.1, problem 10, i.e., let H and K be subgroups of a group G. Show that if $H \cup K$ is a subgroup of G, then either $H \subset K$ or $K \subset H$. Conclude that if $G = H \cup K$ is a union of two subgroups, then one of the subgroups is all of G.

4. Let G be a group and let $x \in G$. Prove that x and x^{-1} have the same order.

5. Let G be a group and let $x \in G$ and suppose x has odd order. Prove that $x = x^{2k}$ for some integer k.

6. Let G be a group and let $x, g \in G$. Show that x and gxg^{-1} have the same order. Prove that for all $a, b \in G$, ab and ba have the same order.

7. Let $G = GL(2, \mathbf{C})$, the group of invertible 2 by 2 complex matrices. Let

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \mathbf{I} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; \mathbf{K} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Let $H = \{\pm\mathbf{1}, \pm\mathbf{I}, \pm\mathbf{J}, \pm\mathbf{K}\}$, where if

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; -X = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.$$

Verify that H is a subgroup, and $\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = -\mathbf{1}$ and $\mathbf{I} \times \mathbf{J} = \mathbf{K}$, $\mathbf{J} \times \mathbf{K} = \mathbf{I}$, and $\mathbf{K} \times \mathbf{I} = \mathbf{J}$. Is H abelian?

8. (a) Let G be a group and let $H_i, i \in I$ be a family of subgroups of G. Prove that the intersection $\cap_{i \in I} H_i$ is a subgroup of G.

(b) Let $G = \mathbf{Z}$ in part (a), and let $H = m\mathbf{Z}$ and $K = n\mathbf{Z}$, and assume that $m \neq 0$ and $n \neq 0$. Show that $H \cap K = [m, n]\mathbf{Z}$, where $[m, n]$ is the least positive integer divisible by both m and n.

(c)* In the setting of part (b), prove that $[m, n](m, n) = mn$.

9. Let G be a group of even order $2n$. Prove there exists $x \in G$ such that the order of x is 2.

10. Let G be the set of complex numbers z such that there is an integer $n_z \in \mathbf{Z}_{>0}$ such that $z^{n_z} = 1$. Prove that G is a group under multiplication of complex numbers. Is G a group under addition of complex numbers? Explain why or why not.

11. Let G be a group and assume that $x^2 = e$ for all $x \in G$. Prove that G is abelian.

12. Let $G = \langle a \rangle$ be a finite cyclic group of order n. Prove that if H is a subgroup of G, then $H = \langle a^k \rangle$ for some k dividing n, so in particular every subgroup of G is cyclic (hint: see how the analogous result was proved for $G = \mathbf{Z}$).