Math 60210, Basic Algebra, Problem Set 10, Fall 2018
due Wed, November 14
Do 8 of these problems

1. Let d be a square-free integer (n^2 does not divide d for $n > 1$ and $d \neq 1$). Prove that $R[x]/(x^2 - d) \cong R[\sqrt{d}]$ for $R = \mathbf{Z}$ and $R = \mathbf{Q}$.

2. Let $\zeta = e^{2\pi i/3}$ and let $R = \mathbf{Z}[\zeta] = \{a + b\zeta : a, b \in \mathbf{Z}\}$, which is a subring of $\mathbf{C}$ by Problem 4 in Problem Set 8. Prove that $\mathbf{Z}[x]/(x^2 + x + 1) \cong \mathbf{Z}[\zeta]$.

3. Let R be an integral domain. Prove that if $R[x]$ is a principal ideal domain, then R is a field (hint: consider the case of $\mathbf{Z}[x]$, and recall what is special about the ideals of a field).

4. Let R be a commutative ring with ideals $I = (\alpha_1, \ldots, \alpha_m)$ and $J = (\beta_1, \ldots, \beta_n)$. Prove that $I \cdot J = (\{\alpha_i\beta_j\}_{i=1,\ldots,m;j=1,\ldots,n})$, i.e., $I \cdot J$ is the ideal generated by all products $\alpha_i\beta_j$.

5. Definition : Let K be a field. A discrete valuation on K is a function $v : K \to \mathbf{R}^{\geq 0}$ satisfying

(1) $v(x) \geq 0$ for all $x \in K$ and $v(x) = 0 \iff x = 0$.

(2) For all $x, y \in K$, $v(xy) = v(x)v(y)$.

(3) For all $x, y \in K$, $v(x + y) \leq max\{v(x), v(y)\}$.

(4) $v(K^*) = \{c^n : n \in \mathbf{Z}\}$ for some real c with $0 < c < 1$.

Let $R_v := \{a \in K | v(a) \leq 1\}$. Prove that R_v is a ring, and $I_v := \{a \in K | v(a) < 1\}$ is an ideal. Choose $\pi \in I_v$ such that $v(\pi)$ is maximal. Prove that I_v is a maximal ideal of R_v, and every ideal of R_v is (π^k) for some $k > 0$. In particular, prove that R_v is a Principal Ideal Domain.

6. Let p be a prime of $\mathbf{Z}$. Let $\alpha \in \mathbf{Q}$ and write $\alpha = p^k \cdot \frac{a}{b}$ with $a, b \in \mathbf{Z}$ and $(p, a) = 1 = (p, b)$. Let $v_p(\alpha) = 2^{-k}$ and $v_p(0) = 0$. Show that v_p is a discrete valuation on $\mathbf{Q}$ and compute R_{v_p} and I_{v_p} for each prime p. Is the field R_{v_p}/I_{v_p} finite?

7. Let $\phi : R \to S$ be a ring homomorphism between commutative rings. Prove that if P is a prime ideal of S, then $\phi^{-1}(P)$ is a prime ideal of R.

8. Let $\phi : R \to S$ be a surjective ring homomorphism and let J be an ideal of S. Prove that $R/\phi^{-1}(J) \cong S/J$.

9.

(i) Let R be an integral domain and let $a \in R$, and let $E_a : R[x] \to R$ be the evaluation homomorphism $E_a(p) = p(a)$. Prove that $R[x]/(x - a) \cong R$, and if $f \in R[x]$, $f \equiv f(a) \pmod{(x - a)}$.

(ii) Let F be a field and let $a_1, \ldots, a_k \in F$ be distinct and let $b_1, \ldots, b_k \in F$. Prove there is $f \in F[x]$ such that $f(a_i) = b_i$ for $i = 1, \ldots, k$ (hint: try to use the Chinese Remainder Theorem for this).

10. Let $R = \mathbf{Z}[\sqrt{d}]$ with d square-free as in problem 1 above, and let $\tau : R \to R$ be the map given by $\tau(a + b\sqrt{d}) = a - b\sqrt{d}$. Prove that τ is a ring automorphism of R.

11. Let $R = \mathbf{Z}[\sqrt{-5}]$. Let

$$I_1 = (2, 1 + \sqrt{-5}), I_3 = (3, 1 + \sqrt{-5}), I_4 = (3, 1 - \sqrt{-5}).$$

Prove that I_1, I_3, and I_4 are maximal ideals of R (hint: the previous problem may make your life easier).

12. Let $R = \mathbf{Z}[\sqrt{-5}]$. Let $I_1 = (2, 1 + \sqrt{-5})$, $I_2 = (2, 1 - \sqrt{-5})$, $I_3 = (3, 1 + \sqrt{-5})$, $I_4 = (3, 1 - \sqrt{-5})$.

(i) Prove that $I_1 = I_2$.

(ii) Prove that $I_1 \cdot I_1 = (2)$, $I_3 \cdot I_4 = (3)$, and $I_1 \cdot I_3 = (1 + \sqrt{-5})$ and $I_1 \cdot I_4 = (1 - \sqrt{-5})$.