Math 60210, Basic Algebra, Problem Set 12, Fall 2018
due Wed, December 5

Do 8 of these problems

1. Let $R = F[x, y]/(xy)$ where F is a field. Let $S = \{x^n : n \geq 0\}$. Prove that $S^{-1}R \cong \{\frac{p(x)}{x^n} : p(x) \in F[x], n \geq 0\}$, which is a subring of the fraction field of $F[x]$ (hint: show $\frac{y^k}{1} = 0$ in $S^{-1}R$).

2. Let F be a field, and consider the integral domain $R = F[[x]]$. Show that the fraction field $Frac(R)$ of R coincides with the subset $\cup_{n\geq 0}\{\frac{a}{x^n} : a \in F[[x]]\}$ of $Frac(R)$ (this problem is repeated from problem set 11, so please don't do it if you did this problem for problem set 11).

3. Let F be a field and let $R = F[x]$. Let $S = \{x^n : n \geq 0\}$. Prove that $S^{-1}R \cong F[x, y]/(xy-1)$.

4. Prove that the following polyomials are irreducible in $\mathbf{Q}[x]$.

(a) $x^4 - 4x^3 + 6$.

(b) $x^4 + 4x^3 + 6x^2 + 2x + 1$.

(c) $x^4 + 10x^2 + 1$.

5. Prove that the following polynomials are irreducible in $\mathbf{C}[x, y]$.

(a) $x^5 + y^2 + 1$.

(b) $x^3 + y^3 + 1$.

(c) $x^{25} - y^{29} - y^{28}$.

6. Let R be an integral domain. Show that if $a \in R$ is nonzero, then $\{a\}$ is a linearly independent subset of the R-module R. Show that $R \cdot a$ is a free R-module. Which elements of $R \cdot a$ give a basis of $R \cdot a$?

7. Let R be an integral domain. Consider the R-module R.

(i) Show that if $a, b \in R$, then the set $\{a, b\}$ is linearly dependent in the R-module R.

(ii) Show that a nonzero ideal I of R is a free R-module if and only if I is principal.

(iii) If $R = \mathbf{Z}[\sqrt{-5}]$ and $M = R$, show that the ideal $(1 + \sqrt{-5}, 2)$ is a submodule of the free module R, but is not free.

(iv) If $R = F[x, y]$ and $M = R$, show that the ideal (x, y) of R is a submodule of the free module R, but is not free.

8. Prove that $\mathbf{Q}$ is not a free $\mathbf{Z}$-module (hint: as in (i) of the previous problem, show that if $a, b \in \mathbf{Q}$, then the set $\{a, b\}$ is linearly dependent over $\mathbf{Z}$).

9. Let R be a ring with left ideal I. Show that R/I is a cyclic R-module.

10. Let $R = \mathbf{Z}$ and consider the R-module $\mathbf{Z}_m \times \mathbf{Z}_n$ with $a \cdot (x, y) = (ax, ay)$ for $x \in \mathbf{Z}_m$ and $y \in \mathbf{Z}_n$. Compute $Ann_{\mathbf{Z}}(1, 1)$ and show that $\mathbf{Z} \cdot (1, 1) \cong \mathbf{Z}_s$ for some integer s. Determine s.

11. Let F be a field and let $R = F[x]$. Let $M = F[x]/(x^2)$ and let $N = F[x]/(x) \oplus F[x]/(x)$. Is $M \cong N$ as a F-vector space? Is $M \cong N$ as a R-module?

12. Let $M_1, \ldots, M_n$ be R-modules. For each $i \in \{1, \ldots, n\}$, let $N_i \subset M_i$ be a submodule.

(i) Show that there is an injective R-module homomorphism $N_1 \oplus \cdots \oplus N_n \to M_1 \oplus \cdots \oplus M_n$, so we may regard $N_1 \oplus \cdots \oplus N_n$ as a submodule of $M_1 \oplus \cdots \oplus M_n$.

(ii) Show that

$$(M_1 \oplus \cdots \oplus M_n)/(N_1 \oplus \cdots \oplus N_n) \cong (M_1/N_1) \oplus \cdots \oplus (M_n/N_n).$$

13. Let V be a vector space over a field F, possibly infinite dimensional. Prove that V has a basis, i.e., a linearly independent subset S that spans V over F (hint: use Zorn's lemma to produce a maximal linearly independent set).