HOMEWORK 2, MATH 60210, BASIC ALGEBRA, DUE WEDNESDAY, SEPT. 5
INSTRUCTOR, SAM EVENS, FALL 2018

INSTRUCTIONS: Do 9 of these 12 problems.

1. Let X be a set and suppose $|X| \geq 3$. Show that $A(X)$, the set of bijections of X, is a nonabelian group (cf. Ash, 1.2, problem 5).

2. (i) Let $\sigma \in S_n$ be a k-cycle. Compute the order of σ.

(ii) Let $\sigma = c_1 \dots c_j$ be a product of j mutually disjoint cycles. Prove that $\sigma^n = e$ if and only if $c_i^n = e$ for all i.

(iii) Suppose in the setting of part (ii) that the element c_i is a n_i cycle. Give a formula for the order of σ in terms of $n_1, \dots, n_j$.

3. (i) Let $c \in S_n$ be a k-cycle. Prove that c is a product of $k-1$ transpositions.

(ii) If $\sigma \in S_n$, show that σ is a product of transpositions.

4. For $i = 1, \dots, n-1$, let τ_i be the transposition $(i,\ i+1)$ of S_n. We call $\tau_1, \dots, \tau_{n-1}$ the simple transpositions of S_n.

(i) Show that $\tau_i \tau_j = \tau_j \tau_i$ if $|j-i| \geq 2$ and $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ for $i = 1, \dots, n-2$ (the second equality is called the braid relation).

(ii) Show that if $\sigma \in S_n$, then σ is a product of simple transpositions.

(iii) Let $\sigma = (1,\ 4)(2,\ 3)$, product of two disjoint transpositions. Show that we can write σ as a product of six simple transpositions.

5. Is there an element of order 6 in S_5? Is there an element of order 9 in S_5? What is the smallest positive integer n such that S_n has an element of order 21?

6. Let $G = D_{2n}$, the symmetries of a regular n-gon. In class, we showed that $|G| = 2n$, and $G = \{r_{2\pi j/n} : j = 0, \dots, n-1\} \cup \{s_{\pi j/n} : j = 0, \dots, n-1\}$, where r_θ is counterclockwise rotation by an angle of θ and s_θ is reflection through a line through the origin at angle θ to the x-axis. Let $r = r_{2\pi/n}$ and $s = s_0$.

(i) Prove that $G = \{r^j : j = 0, \dots, n-1\} \cup \{r^j s : j = 0, \dots, n-1\}$, and find an angle θ such that $r^j s = s_\theta$.

(ii) Show that $r^n = e$, $s^2 = e$, and $sr = r^{-1}s$ and G is abelian if and only if $n = 1$ or 2.

7. Let F be a field, let $n \in \mathbf{Z}_{>0}$, and let $M(n, F)$ be the n by n matrices with entries in F. For $A \in M(n, F)$, let A^t be the transpose of A. Let

$$O(n, F) := \{A \in GL(n, F) : A^t = A^{-1}\} = \{A \in GL(n, F) : A^t \times A = I_n\}.$$

(i) Prove that $O(n, F)$ is a subgroup of $GL(n, F)$.

(ii) Prove that if $A \in O(2) = O(2, \mathbf{R})$ and $\det(A) = 1$, then

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

for some $\theta \in \mathbf{R}$. Prove that if $A \in O(2)$ and $\det(A) = -1$, then

$$A = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ \sin(\phi) & -\cos(\phi) \end{pmatrix}$$

for some $\phi \in \mathbf{R}$.

8. Let H be a subgroup of a group G. Give an explicit bijection between the set G/H of left cosets G/H and the set $H\backslash G$ of right cosets.

9. Let H be a subgroup of a group G of index 2 (i.e., $[G : H] = 2$). Prove that H is a normal subgroup of G.

10. Let G be a group, and let the center $Z(G)$ be the set $\{z \in G : gz = zg \ \forall g \in G\}$.

(i) Prove that $Z(G)$ is a normal subgroup of G.

(ii) Show that the center of S_n is $\{e\}$ for $n \geq 3$.

(iii) Find the center of D_{2n}.

11. Prove that the center of $GL(n, F)$ is the set of scalar matrices $\{\lambda I_n : \lambda \in F^\times\}$.

12*. Let G be a group and let H and K be two subgroups of G of finite index, i.e., $[G : H]$ and $[G : K]$ are finite. Prove that $H \cap K$ is a subgroup of finite index in G.