HOMEWORK 3, MATH 60210, BASIC ALGEBRA
DUE WEDNESDAY, SEPT. 12, FALL 2018

INSTRUCTIONS: Do 9 of these 12 problems.

1. Prove that if G, H, and K are groups, then

(i) $G \cong G$

(ii) If $G \cong H$, then $H \cong G$.

(iii) If $G \cong H$ and $H \cong K$, then $G \cong K$.

2. Let F be a field, and let

$$N(2, F) := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in F \right\}.$$

Prove that $N(2, F)$ is a subgroup of $GL(2, F)$ and $(F, +) \cong N(2, F)$.

3. Let $\phi : G \to H$ be a group homomorphism, and let $A \subset G$ be a cyclic subgroup, i.e., $A = \langle a \rangle$ for some $a \in G$. Prove that $\phi(A)$ is a cyclic subgroup of H.

4. Let $\phi : G \to H$ be a group homomorphism and let $a \in G$. Prove that $|\phi(a)|$ divides $|a|$ if $|a|$ is finite.

5. Recall the group $\{Id_2, I, J, K, -Id_2, -I, -J, -K\} \subset GL(2, \mathbf{C})$ from Problem Set 1, problem 7. In the sequel, we will call this group Q_8 (it is based on Quaternions, and has 8 elements). Is $Q_8 \cong D_8$? Justify your answer.

6. For a group G, let $\mathrm{Int}(G) = \{c_x : x \in G\}$, where $c_x(g) = xgx^{-1}$. Let $\mathrm{Aut}(G)$ is the set of all isomorphisms $\phi : G \to G$.

(a) Prove that the map $\psi : G \to \mathrm{Aut}(G)$ given by $\psi(x) = c_x$ is a group homomorphism with image $\mathrm{Int}(G)$.

(b) Prove that $\mathrm{Int}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$, and $\mathrm{Int}(G) \cong G/Z(G)$, where $Z(G)$ is the center of G.

7. (see D+F, problem 18 of 3.2): Let G be a finite group and let H and N be subgroups of G with N normal in G. Prove that if $|H|$ and $[G : N]$ are relatively prime, then $H \subset N$.

8. Prove that $\mathbf{C}/\mathbf{Z} \cong \mathbf{C}^\times$, where $\mathbf{Z}$ is regarded as a subset of $\mathbf{C}$ via $n = n + 0i$ for $n \in \mathbf{Z}$.

9. Let $G = D_{2n}$, so G is the group generated by a rotation $r = r_{2\pi/n}$ and a reflection s, where $s(x, y) = (x, -y)$. Let k be a divisor of n, and let $H_k := \langle r^k \rangle = \{r^{jk} : j = 0, \ldots, \frac{n}{k} - 1\}$. Prove that H_k is a normal subgroup of G. Do you think G/H_k is a dihedral group? If so, which one?

10. (cf. D+F, problem 6) Show that $\mathbf{R}^\times/\{\pm 1\} \cong \mathbf{R}^{>0}$, the group of positive real numbers under multiplication.

11. Let n be an integer with $n \geq 2$. Prove that $\mathrm{Aut}(\mathbf{Z}/n\mathbf{Z}) \cong (\mathbf{Z}/n\mathbf{Z})^\times$.

12. In class, for sets S, T, we define $\mathrm{Map}(S, T)$ to be the collection of all maps $f : S \to T$. If R is another set, and $\phi : R \to S$ is a map, then we define $\phi^* : \mathrm{Map}(S, T) \to \mathrm{Map}(R, T)$ by $\phi^*(f) = f \circ \phi$.

(i) If $\psi : Q \to R$ is a map of sets, then show that $(\phi \circ \psi)^*(f) = \psi^*(\phi^*(f))$.

(ii) If $\phi : R \to S$ is bijective, prove that $\phi^* : \mathrm{Map}(S, T) \to \mathrm{Map}(R, T)$ is bijective.

(iii) Let $R = \mathbf{R}^3$ and $S = \mathbf{R}^2$. Let $T = \mathbf{R}$ and note that $\mathrm{Map}(R, T)$ and $\mathrm{Map}(S, T)$ are functions on $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively. Recall that if S is a set, and $f, g \in \mathrm{Map}(S, \mathbf{R})$ are functions, we define functions $(f + g)$ and $(f \cdot g)$ in $\mathrm{Map}(S, \mathbf{R})$ by $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$. Denote a point of $\mathbf{R}^3$ by (a_1, a_2, a_3), and let x_1, x_2, x_3 be the functions on $\mathbf{R}^3$ given by $x_i(a_1, a_2, a_3) = a_i$ for $i = 1, 2, 3$. Denote a point of $\mathbf{R}^2$ by (b_1, b_2) and let y_1, y_2 be the functions on $\mathbf{R}^2$ given by $y_i(b_1, b_2) = b_i$ for $i = 1, 2$. Let $\phi : \mathbf{R}^3 \to \mathbf{R}^2$ be given by $\phi(a_1, a_2, a_3) = (a_1 a_2 + a_3^2, a_2 a_3)$. Compute $\phi^*(y_1)$ and $\phi^*(y_2)$ as functions on $\mathbf{R}^3$.