Math 60210, Basic Algebra, Problem Set 4, Fall 2018
due Wednesday, September 19
Do 8 of these problems

1. Let X and Y be two sets. Suppose there exists a bijection $\phi : X \to Y$. Prove that $A(X) \cong A(Y)$, where $A(X)$ and $A(Y)$ are the bijections of X and Y.

2. Let G be a group with subgroup H. As usual, for $g \in G$, let $gHg^{-1} = \{gxg^{-1} : x \in H\}$. Let $N_G(H) = \{g \in G : gHg^{-1} = H\}$. $N_G(H)$ is called the normalizer of H in G. Prove

(i) $N_G(H)$ is a subgroup of G.

(ii) $H \subset N_G(H)$ and H is normal in $N_G(H)$.

(iii) Let $K \subset G$ be a subgroup such that $H \subset K$ and H is normal in K. Prove that $K \subset N_G(H)$, so that $N_G(H)$ is the maximum subgroup of G (with respect to inclusion) containing H with H a normal subgroup.

3. For $k \in \mathbf{Z}_{>0}$, let C_k be a cyclic group of order k. Let $m, n \in \mathbf{Z}_{>0}$. If the greatest common divisor $(m, n) > 1$, then prove that $C_m \times C_n$ is not cyclic.

4. Recall that if G and H are groups, then $\mathrm{Hom}(G, H)$ is the set of all group homomorphisms from G to H.

(i) For any group H, prove that the map $\phi \mapsto \phi(1)$ is a bijection $\mathrm{Hom}(\mathbf{Z}, H) \to H$.

(ii) Let $G = \langle a \rangle$ be a cyclic with a an element of order a. Prove that the map $\phi \mapsto \phi(a)$ is a bijection $\mathrm{Hom}(G, H) \to \{x \in H : x^n = e\}$.

5. Let H and K be groups. Let $\sigma : K \to \mathrm{Aut}(H)$ be a homomorphism from K to the automorphism group of H. Let $H \ltimes_\sigma K$ be the set $\{(x, y) : x \in H, y \in K\}$. Define a product on $H \ltimes_\sigma K$ by the formula

$$(x, y) \cdot (u, v) = (x \cdot \sigma(y)(u), y \cdot v),$$

for $x, u \in H$ and $y, v \in K$. Prove that $H \ltimes_\sigma K$ with this product is a group with identity (e_H, e_K) where e_H, e_K are the identity elements of H and K respectively.

6. Let G be a group with normal subgroup H and subgroup K. For $z \in K$, define $c_z : H \to H$ by $c_z(x) = zxz^{-1}$.

(i) Prove that $c_z \in \mathrm{Aut}(H)$, the automorphism group of H, and $\phi : K \to Aut(H)$ given by $\phi(z) = c_z$ is a group homomorphism.

(ii) Define $\chi : H \ltimes_\phi K \to G$ by the formula $\chi(x, y) = x \cdot y$, using the product in G. If $H \cap K = \{e\}$, then prove that χ is an injective group homomorphism.

7. For $n \in \mathbf{Z}_{>1}$, let $\mathbf{Z}_n := \mathbf{Z}/n\mathbf{Z}$, and $\mathbf{Z}_n^\times := (\mathbf{Z}/n\mathbf{Z})^\times$.

(i) Show that $\mathbf{Z}_7^\times$ is a cyclic group of order 6.

(ii) For $m = 2, 3$ construct a group homomorphism $\phi : \mathbf{Z}_m \to \mathrm{Aut}(\mathbf{Z}_7) \cong \mathbf{Z}_7^\times$ (hint: problem 4 may be useful).

(iii) Construct nonabelian groups of order 14 and 21.

8. Let H, K be groups and let $G = H \times K$. Let $A = \{(x, e) : x \in H\}$ and let $B = \{(e, y) : y \in K\}$. We stated in class that $A \cong H$ and $B \cong K$. Prove that A and B are normal subgroups of G and $G/A \cong K$ and $G/B \cong H$.

9. If H and K are groups, prove that $H \times K \cong K \times H$.

10. Prove that $S_n =< \sigma, \tau >$, the group generated by σ and τ, where $\sigma = (1, 2, 3, \ldots, n)$ is the n-cycle and $\tau = (1, 2)$.

11. (a) Let G be a group with normal subgroup N and let H be a subgroup of G. Let $\pi : G \to G/N$ be the quotient homomorphism where $\pi(a) = aN$ for $a \in G$. Prove that $\pi(H) = HN/N$.

(b) Let n be a positive integer, and in part (a), let $G = \mathbf{Z}$ and let $N = n\mathbf{Z}$. If $H = m\mathbf{Z}$ with m a positive integer, prove that $\pi(H) \cong (m, n)\mathbf{Z}/n\mathbf{Z}$.

12. (i) Let G be a group with center $Z(G)$. If $G/Z(G)$ is cyclic, then prove that G is abelian.

(ii) Give an example of a group G such that $G/Z(G)$ is abelian, but G is not abelian.