1. Let $X$ and $Y$ be two sets. Suppose there exists a bijection $\phi : X \to Y$. Prove that $A(X) \cong A(Y)$, where $A(X)$ and $A(Y)$ are the bijections of $X$ and $Y$.

2. Let $G$ be a group with subgroup $H$. As usual, for $g \in G$, let $gHg^{-1} = \{gxg^{-1} : x \in H\}$. Let $N_G(H) = \{g \in G : gHg^{-1} = H\}$. $N_G(H)$ is called the normalizer of $H$ in $G$. Prove
   (i) $N_G(H)$ is a subgroup of $G$.
   (ii) $H \subset N_G(H)$ and $H$ is normal in $N_G(H)$.
   (iii) Let $K \subset G$ be a subgroup such that $H \subset K$ and $H$ is normal in $K$. Prove that $K \subset N_G(H)$, so that $N_G(H)$ is the maximum subgroup of $G$ (with respect to inclusion) containing $H$ with $H$ a normal subgroup.

3. For $k \in \mathbb{Z}_{>0}$, let $C_k$ be a cyclic group of order $k$. Let $m, n \in \mathbb{Z}_{>0}$. If the greatest common divisor $(m, n) > 1$, then prove that $C_m \times C_n$ is not cyclic.

4. Recall that if $G$ and $H$ are groups, then $\text{Hom}(G, H)$ is the set of all group homomorphisms from $G$ to $H$.
   (i) For any group $H$, prove that the map $\phi \mapsto \phi(1)$ is a bijection $\text{Hom}(\mathbb{Z}, H) \to H$.
   (ii) Let $G = \langle a \rangle$ be a cyclic with $a$ an element of order $a$. Prove that the map $\phi \mapsto \phi(a)$ is a bijection $\text{Hom}(G, H) \to \{x \in H : x^n = e\}$.

5. Let $H$ and $K$ be groups. Let $\sigma : K \to \text{Aut}(H)$ be a homomorphism from $K$ to the automorphism group of $H$. Let $H \ltimes_{\sigma} K$ be the set $\{(x, y) : x \in H, y \in K\}$. Define a product on $H \ltimes_{\sigma} K$ by the formula
   $$(x, y) \cdot (u, v) = (x \cdot \sigma(y)(u), y \cdot v),$$
   for $x, u \in H$ and $y, v \in K$. Prove that $H \ltimes_{\sigma} K$ with this product is a group with identity $(e_H, e_K)$ where $e_H, e_K$ are the identity elements of $H$ and $K$ respectively.

6. Let $G$ be a group with normal subgroup $H$ and subgroup $K$. For $z \in K$, define $c_z : H \to H$ by $c_z(x) = zxz^{-1}$.
   (i) Prove that $c_z \in \text{Aut}(H)$, the automorphism group of $H$, and $\phi : K \to \text{Aut}(H)$ given by $\phi(z) = c_z$ is a group homomorphism.
   (ii) Define $\chi : H \ltimes_{\phi} K \to G$ by the formula $\chi(x, y) = x \cdot y$, using the product in $G$. If $H \cap K = \{e\}$, then prove that $\chi$ is an injective group homomorphism.

7. For $n \in \mathbb{Z}_{>1}$, let $Z_n := \mathbb{Z}/n\mathbb{Z}$, and $Z_n^\times := (\mathbb{Z}/n\mathbb{Z})^\times$.
   (i) Show that $Z_n^\times$ is a cyclic group of order 6.
   (ii) For $m = 2, 3$ construct a group homomorphism $\phi : Z_m \to \text{Aut}(Z_7) \cong Z_7^\times$ (hint: problem 4 may be useful).
   (iii) Construct nonabelian groups of order 14 and 21.

8. Let $H, K$ be groups and let $G = H \times K$. Let $A = \{(x, e) : x \in H\}$ and let $B = \{(e, y) : y \in K\}$. We stated in class that $A \cong H$ and $B \cong K$. Prove that $A$ and $B$ are normal subgroups of $G$ and $G/A \cong K$ and $G/B \cong H$.

9. If $H$ and $K$ are groups, prove that $H \times K \cong K \times H$.

10. Prove that $S_n = <\sigma, \tau>$, the group generated by $\sigma$ and $\tau$, where $\sigma = (1, 2, 3, \ldots, n)$ is the $n$-cycle and $\tau = (1, 2)$.
11. (a) Let $G$ be a group with normal subgroup $N$ and let $H$ be a subgroup of $G$. Let $\pi : G \rightarrow G/N$ be the quotient homomorphism where $\pi(a) = aN$ for $a \in G$. Prove that $\pi(H) = HN/N$.

(b) Let $n$ be a positive integer, and in part (a), let $G = \mathbb{Z}$ and let $N = n\mathbb{Z}$. If $H = m\mathbb{Z}$ with $m$ a positive integer, prove that $\pi(H) \cong (m, n)\mathbb{Z}/n\mathbb{Z}$.

12. (i) Let $G$ be a group with center $Z(G)$. If $G/Z(G)$ is cyclic, then prove that $G$ is abelian.

(ii) Give an example of a group $G$ such that $G/Z(G)$ is abelian, but $G$ is not abelian.