**Math 60210, Basic Algebra, Problem Set 5, Fall 2013**
**due Fri, October 5**

Do 6 of these 9 problems, and I would like everyone to do problems 1, 2, and 8.

**1.** Let $G = A_n$, the alternating group in $S_n$. Let $H$ be the subgroup of $A_n$ generated by all 3-cycles. Prove that $H = A_n$ (hint: $A_n$ is generated by products $\tau_1 \circ \tau_2$, where $\tau_1, \tau_2$ are transpositions in $S_n$. Recall that for $\sigma \in S_n$, $Supp(\sigma) = \{i : \sigma(i) \neq i\}$. Show that if $Supp(\tau_1)$ intersects $Supp(\tau_2)$, then $\tau_1 \circ \tau_2$ is either the identity or a 3-cycle. If $Supp(\tau_1)$ does not intersect $Supp(\tau_2)$, show that $\tau_1 \circ \tau_2$ is a product of two 3-cycles. For example, $(1,2) \circ (3,4) = (1,2,3) \circ (2,3,4)$. Conclude that generators of $A_n$ are in $H$).

**2.** Let $n \geq 5$. Let $g, h$ be 3-cycles, and hence in $A_n$. Prove that $g$ is conjugate to $h$ in $A_n$ (hint: we have seen that $g$ is conjugate to $h$ in $S_n$. Let $g = (1,2,3)$ and let $h = (2,1,3)$. Then if $\tau = (1,2)$, then $\tau g \tau^{-1} = h$, but $\tau \notin A_n$. Find a way to adjust $\tau$ to an element $\sigma \in A_n$ so $\sigma g \sigma^{-1} = h$. We really need $n \geq 5$. $g$ is not conjugate to $h$ in $A_4$).

**3.** Determine the conjugacy classes in the dihedral group $G = D_{2p}$, where $p$ is an odd prime. For each conjugacy class $C$, compute $C_G(x)$ for some $x \in C$.

**4.** Determine the conjugacy classes in $G = D_{24}$. For each conjugacy class $C$ in $G$, compute $C_G(x)$ for some $x \in C$.

**5.** Let $V = \mathbf{R}^n$, and let $G = O(n, \mathbf{R}) = \{g \in GL(n, \mathbf{R}) : g^{tr} = g^{-1}\}$. Determine the $G$-orbits on $V$. Show $O(n-1, \mathbf{R}) \cong H := \{g \in G : g \cdot e_n = e_n\}$. Construct a bijection $O(n, \mathbf{R})/H \to S^{n-1}$, where $S^{n-1} = \{v \in V : v \cdot v = 1\}$ is the usual $n-1$-dimensional sphere.

**6.** Let $F$ be a field with 3 elements. Show that $|SL(2,F)| = |S_4|$. Is $SL(2,F) \cong S_4$? Explain your answer.

**7.** Let $G = S_4$. Consider the subgroup

$$V = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

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(i) Prove that $V$ is a normal subgroup of $G$.

(ii) Prove that $S_4/V \cong S_3$.

**8.** Let $G$ be a finite group and let $p$ be the smallest prime dividing $|G|$. Let $H$ be a subgroup of $G$ and suppose $[G : H] = p$. Prove that $H$ is a normal subgroup of $G$ (note: when $p$=2, this problem is the same as problem 9 of Problem Set 2. When $p = 3$, this problem is the same as the second problem on the exam, and the hints there may be helpful).

**9.** Find at least one 2-Sylow subgroup of each of $S_3$, $S_4$, $S_5$, and $S_6$. For extra credit, do the same problem for $S_8$. You do not need to find more than one.