Math 60210, Basic Algebra, Problem Set 6, Fall 2018
due Wed, October 10

Do 8 of these 13 problems. Note that two of the problems counts as two problems. Problem 2-3 would be useful to do, and I encourage everyone to do either Problem 6 or Problem 7-8; Problem 6 is the easier route.

1. Let $F = \mathbf{Z}/p\mathbf{Z}$. Determine the number of p-Sylow subgroups of $SL(2, F)$.

2-3. Show that if G is a nonabelian group of order less than 60, then G is not simple.

4. Let G be a group of order 189.

(i) Prove that G has a normal subgroup of order 3^k for some $k > 0$.

(ii) Prove that G has a normal 7-Sylow subgroup.

5. Prove that the quaternion group Q_8 is not isomorphic to a subgroup of S_4 (hint: use the Sylow theorems).

6. Let $G = A_5$. Let N be a nontrivial normal subgroup of G, and let $\sigma \in N - \{e\}$.

(i) After relabelling the indices, we may assume that $\sigma = (1,\ 2,\ 3)$ or $\sigma = (1,\ 2)(3,\ 4)$, or $\sigma = (1,\ 2,\ 3,\ 4,\ 5)$, i.e., is either a 3-cycle, a product of two transpositions, or a 5-cycle.

(ii) Suppose $\sigma = (1,\ 2)(3,\ 4)$. Let $\tau = (1,\ 2,\ 5)$. Show that $\tau\sigma\tau^{-1}\sigma$ is a 3-cycle in N.

(iii) Suppose $\sigma = (1,\ 2,\ 3,\ 4,\ 5)$ is a 5-cycle. Let $\tau = (1,\ 2,\ 3)$. Show that $\tau\sigma\tau^{-1}\sigma^{-1}$ is a 3-cycle in N.

(iv) Prove that N contains a 3-cycle, and prove that $N = A_5$ (hint: use problems from Problem Set 5.) Conclude that A_5 is simple.

7-8. Let G be a group of order 60 and let n_5 be the number of 5-Sylow subgroups. Prove that if $n_5 > 1$, then G is a simple group. Prove that A_5 is simple (you can use the following series of hints):

(A) Show that $n_5 = 6$.

(B) Let P be a 5-Sylow subgroup. Show that $|N_G(P)| = 10$ (use proof of Sylow (2) and (3)).

(C) Let $H \subset G$ be a normal subgroup and suppose 5 divides $|H|$. Prove that all six 5-Sylow subgroups of G are contained in H.

(D) Let H be as in (C). Prove that if H is a proper normal subgroup, then $|H| = 30$.

(E) Let H be as in (D). Recall that H has a normal cyclic subgroup A of order 15, and show that A has a unique subgroup P of order 5. Conclude that P is normal in H, and use (C) or another step to find a contradiction.

(F) Suppose G has a normal subgroup H of order 2, 3 or 4. Prove that G/H has a normal 5-Sylow subgroup, and use this to prove that G has a proper normal subgroup of order divisible by 5, contradicting (E). Use the correspondence theorem for this, and for one of these cases, you may want to recall that a group of order 30 has a unique 5-Sylow subgroup.

(G) Let G have a normal subgroup H of order 6 or 12. Prove that H has a unique 2-Sylow subgroup or 3-Sylow subgroup P. Prove that P is normal in G, and apply step (F).

9. For each n from the list below, explain why no group of order n can be simple.

(a) $n = 2^4 \cdot 5^6$.

(b) $n = 187$.

(c) $n = 343$.

10. Let G be a simple group of order 168. Determine the number of elements of order 7 in G.

11. Let $G = \mathbf{Z}/28\mathbf{Z}$. Find two different composition series for G and verify that the composition factors are the same up to reordering.

12. (A) Let $G = D_{56}$. Find a composition series for G.

(B) Explain how to find a composition series for D_{2n} for any integer n (this can be done in rough terms, i.e., just give the idea).

13. Let $F = \mathbf{Z}/3\mathbf{Z}$. Give a composition series for $GL(2, F)$.