Math 60210, Basic Algebra, Problem Set 7, Fall 2018
due Wed, October 24

Try to do all seven of these problems. Your professor will be tolerant if you decide it is fall break, and the "break" part of fall break is important to you.

Most of these problems use the following notation. Let F be a field, and let $n \in \mathbf{Z}_{\geq 1}$. Let $e_1, \ldots, e_n$ be the standard basis of $V = F^n$, and for $i = 1, \ldots, n$, let F_i be the span of $\{e_1, \ldots, e_i\}$. Note that F_i is a i-dimensional subspace of V, and hence defines a point in $\mathrm{Gr}(i, V)$, the set of all i-dimensional subspaces of V. Let $x = (F_1, \ldots, F_n)$, which is a point in $X := \prod_{i=1}^n \mathrm{Gr}(i, V)$. Let $F_j = 0$ for $j \leq 0$. Then $GL(n, F)$ acts on X via its action on each factor. Let

$$B(n, F) = \{A = (a_{ij}) \in GL(n, F) : a_{ij} = 0 \ \forall \ i > j\}$$

be the group of upper triangular invertible matrices, and $B(n, F)$ is the stabilizer in $GL(n, F)$ of the point x_0 (this is verified in notes on the course webpage titled: " Notes on subgroups of the general linear group ...", and you do not need to verify this yourself). For each integer $k > 0$, let

$$\mathfrak{n}_k = \{A = (a_{ij}) \in M(n, F) : a_{ij} = 0 \ \forall \ j - i < k\}.$$

Note that $\mathfrak{n}_k$ is the span of all elementary matrices E_{ij} with $j - i \geq k$. We let $N_k = \{Id_n + X : X \in \mathfrak{n}_k\}$. Note that N_1 is the group we have called $N(n, F)$.

1. Show that $\mathfrak{n}_k = \{A \in M(n, F) : A(F_i) \subset F_{i-k}\}$ and if $A \in \mathfrak{n}_k, B \in \mathfrak{n}_t$, then the product $AB \in \mathfrak{n}_{k+t}$. If $A \in \mathfrak{n}_k$ and $B \in B(n, F)$, then BA and AB are in $\mathfrak{n}_k$.

2. Show that N_k is a subgroup of $B(n, F)$ (hint: observe that $X^n = 0$ for $X \in \mathfrak{n}_k$, and find the inverse of $Id_n + X$; alternatively look at how we proved N_1 is a subgroup in class).

3. Show that if $g \in B(n, F)$, then $g = D \cdot U$, where D is an invertible diagonal matrix and $U \in N_1$ (hint: if A is an upper triangular matrix, then $A = D + X$, where D is diagonal and $X \in \mathfrak{n}_1$).

4. Show that the commutator subgroup $[N_k, N_k] \subset N_{k+1}$ (hint: write out the commutator explicitly).

5. Show that $[B(n, F), B(n, F)] \subset N_1$ (hint: use the decomposition of g in Problem 3), and N_k is a normal subgroup of $B(n, F)$. Conclude that N_{k+1} is a normal subgroup of N_k, and show N_k/N_{k+1} is abelian.

6. Let k be a positive integer. Prove that the kth commutator subgroup $B(n, F)^{(k)} \subset N_k$ and $B(n, F)$ is solvable. Prove that $[N_1, N_k] \subset N_{k+1}$ and $N_1 = N(n, F)$ is nilpotent.

7. Let

$$G_0 \stackrel{\phi_0}{\to} G_1 \stackrel{\phi_1}{\to} \cdots \stackrel{\phi_{n-2}}{\to} G_{n-1} \stackrel{\phi_{n-1}}{\to} G_n \ldots$$

be a sequence of groups and maps. Recall that we call such a sequence exact if for all i, the image of ϕ_i equals the kernel of ϕ_{i+1}. Let 1 denote the trivial group $\{e\}$, and for any group H, let $1 \to H$ and $H \to 1$ be the unique group homomorphisms from the trivial group to H and from H to the trivial group. By convention, we do not label these arrows.

(i) Let $1 \to A \stackrel{\phi}{\to} B$ be a sequence of groups and group homomorphisms. Prove that ϕ is injective if and only if the sequence is exact.

(ii) Let $A \stackrel{\phi}{\to} B \to 1$ be a sequence of groups and group homomorphisms. Prove that ϕ is surjective if and only if the sequence is exact.

(iii) Let $1 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 1$ be a sequence of groups and group homomorphisms. Prove the sequence is exact if and only if ϕ is injective, ψ is surjective, and $B/\phi(A) \cong C$ by the map $\overline{\psi}(x\phi(A)) = \psi(x)$ for $x \in B$.