Math 60210, Basic Algebra, Problem Set 8, Fall 2018  
Due Wed, October 31

Do 8 of these 12 problems. Everyone should read 5, 9, and 10 and remember the notation. As in class, \( R[x] \) is the polynomial ring with coefficients in \( R \).

1. (A) Let \( R \) be a ring. An element \( x \in R \) is called nilpotent if \( x^n = 0 \) for some \( n > 0 \). If \( x \in R \) is nilpotent, prove that \( 1 + x \in R^\neq \).

(B) If \( F \) is a field and \( A \in M(n, F) \) has the property that the entries \( A_{i,j} = 0 \) for \( i \geq j \), prove that \( (I + A) \) is invertible and give a formula for \( (I + A)^{-1} \).

2. Let \( d \) be a square-free integer, i.e., \( d \neq 1 \) and \( m^2 \) does not divide \( d \) for every integer \( m \geq 2 \). Let \( E = \mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Q}\} \). Prove that \( E \) is a subring of \( \mathbb{C} \). Is \( E \) a field? Explain why or why not.

3. Let \( d \) be a square-free integer not equal to 1. Let \( R = \mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\} \).
   (i) Prove that \( R \) is a subring of \( \mathbb{C} \).
   (ii) Let \( N(\alpha) = \alpha \cdot \tau(\alpha) \), where for \( \alpha = a + b\sqrt{d} \in R \), \( \tau(\alpha) = a - b\sqrt{d} \). Prove that \( N(\alpha) \in \mathbb{Z} \), and \( N(\alpha \cdot \beta) = N(\alpha) \cdot N(\beta) \) for all \( \alpha, \beta \in R \).
   (iii) Let \( d < 0 \). Compute the unit group \( R^\neq \) of \( R \) if \( d < 0 \) (hint: show \( \alpha \in R^\neq \) if and only if \( N(\alpha) = 1 \)).

4. Let \( \zeta = e^{2\pi i/3} \). Let \( R = \{a + b\zeta : a, b \in \mathbb{Z}\} \). Prove that \( R \) is a subring of \( \mathbb{C} \) and compute the unit group \( R^\neq \) of \( R \) if \( d < 0 \) (hint: show \( \alpha \in R^\neq \) if and only if \( N(\alpha) = 1 \)).

5. Let \( R \) be a ring, and let \( R^\times \) be the set of all units of \( R \). Prove that \( R^\times \) is a group under multiplication.

6. (i) Let \( R \) be an integral domain. Prove that the unit group \( R[x]^\neq \) of \( R[x] \) is \( R^\times \).
   (ii) Let \( R \) be a ring with nonzero (nilpotent) element \( a \) with \( a^n = 0 \) for some \( n \). Prove that \( 1 - ax \) is a unit of \( R[x] \).

7. Let \( F \) be a field. Prove that if \( f = \sum_{i=0}^\infty a_i x^i \in F[[x]] \), then \( f \) is a unit if and only if \( a_0 \neq 0 \). Can you find the unit group of \( R[[x]]^\times \) for an integral domain \( R \)?

8. Let \( H \) be the quaternions, i.e.,
   \[
   H = \{A \in M(2, \mathbb{C}) : A = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \},
   \]
   and for \( A \) as above, we call \( \alpha \) and \( \beta \) the coordinates of \( A \). We said in class that \( H \) is a subring of the two by two complex matrices \( M(2, \mathbb{C}) \), and \( H \) is a real subspace of \( M(2, \mathbb{C}) \) with basis \( 1, I, J, K \), where \( 1 \) is the 2 by 2 identity matrix,
   \[
   I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
   \]
   We defined \( N(A) = \frac{1}{2} \text{Tr}(A \cdot -A^*) = |\alpha|^2 + |\beta|^2 \) for \( A \) with coordinates \( \alpha, \beta \).
   (i) For \( A, B \in H \), prove that \( N(A \cdot B) = N(A) \cdot N(B) \).
   (ii) Let \( H_1 := \{A \in H : N(A) = 1 \} \). Prove that \( H_1 \) is a group using the multiplication in \( H \) (the group \( H_1 \) is often called \( SU(2) \), the determinant one unitary 2 by 2 matrices).
   (iii) Prove that center \( Z(H) = R \cdot 1 \), i.e., the real multiples of the identity matrix.

9. Let \( R \) be a ring and let \( a \in R \). Let \( C_R(a) = \{x \in R : xa = ax\} \). Prove that \( C_R(a) \) is a subring of \( R \).
10. Let $R$ be a ring and let $\{S_i\}_{i \in I}$ be a family of subrings of $R$. Prove that $\cap_{i \in I} S_i$ is a subring of $R$. Let $Z(R) = \{x \in R : xy = yx, \forall y \in R\}$ ($Z(R)$ is called the center of $R$). Show $Z(R) = \cap_{a \in R} C_R(a)$, so $Z(R)$ is a subring of $R$.

11. Let $F$ be the free group on a set $S$ with two elements and let $x$ and $y$ be the generators of $F$ corresponding to elements of $S$. Fix $n > 0$, and let $A(n) = F/N$, where $N$ is the smallest normal subgroup of $F$ containing $x^n, y^2,$ and $yxy^{-1}x$.

(i) Let $a = xN$ and let $b = yN$. Prove that every element of $A(n)$ can be written in the form $a^ib^j$, where $i = 0, \ldots, n - 1$ and $j = 0, 1$, and conclude that $|A(n)| \leq 2n$.

(ii) Prove that $A(n) \cong D_{2n}$ (hint: there is a surjective homomorphism from $F$ to $D_{2n}$).

12. Let $R = M(n, F)$, where $F$ is a field. Let $I$ be a two-sided ideal of $R$. Prove that either $I = 0$ or $I = R$ (hint: $aI \subset I$ and $Ia \subset I$ for every $a \in R$. Use elementary matrices).