Math 60210, Basic Algebra, Problem Set 9, Fall 2018
due Wed, November 7
Do 8 of these problems

1. Let $R = \{a + b\sqrt{-5} : a, b \in \mathbf{Z}\}$. If $\alpha \in R$ is nonzero, prove that $R/(\alpha)$ is finite. Conclude that if $I \subset R$ is a nonzero ideal, then R/I is finite.

2. Let $R = \mathbf{Z}[x]$. Let $I = (2, x)$ be the ideal of R generated by I and x. Prove that I is not a principal ideal (f) for any $f \in R$.

3. Let $R = \mathbf{Z}[\sqrt{-5}]$ as in problem 1. Let $I = (2, 1 + \sqrt{-5})$ be the ideal of R generated by 2 and $1 + \sqrt{-5}$. Prove that I is not a principal ideal of R, i.e., $I \neq (\alpha)$ for any $\alpha \in R$ (hint: if $I = (\alpha)$, then $2, 1 + \sqrt{-5} \in (\alpha)$, so $2 = \beta \cdot \alpha$ for some $\alpha \in R$. What does this say about $N(2)$ and $N(\alpha)$? Argue similarly with $1 + \sqrt{-5}$ in place of 2).

4. Let $R = \{a + b\sqrt{-5} : a, b \in \mathbf{Z}\}$. R is a subring of the integers. Consider the ideal $I = (2, 1 + \sqrt{-5})$. Prove that $R/I = \{0 + I, 1 + I\}$ is a field with 2 elements.

5. Let R_1 and R_2 be rings. Prove that $(R_1 \times R_2)^* = R_1^* \times R_2^*$. Prove that if m and n are relatively prime integers, then $\phi(mn) = \phi(m)\phi(n)$, where $\phi(k) = |\mathbf{Z}_k^*|$ (hint: use the Chinese Remainder Theorem to relate $\mathbf{Z}_{mn}$ to $\mathbf{Z}_m$ and $\mathbf{Z}_n$.) Compute $\phi(p^n)$ for p prime, and give a formula for $\phi(n)$ in terms of the prime factorization of n.

6. Let R, S, T be rings.

(i) Prove $R \cong R$.

(ii) If $R \cong S$, prove that $S \cong R$.

(iii) If $R \cong S$ and $S \cong T$, prove that $R \cong T$.

7. Let R be a commutative ring with ideals I and J. Prove $I \cdot J \subset I \cap J$. If $R = I + J$, prove that $I \cap J = I \cdot J$.

8. Let S be a finite set, and let $\mathcal{P}(S)$ be the set of all subsets of S. For $T, U \in \mathcal{P}(S)$, define $T + U := T \cup U - T \cap U$. Define $T \cdot U := T \cap U$. Show that $\mathcal{P}(S)$ is a commutative ring with the operations $(+, \cdot)$ and find the characteristic of $\mathcal{P}(S)$.

9. Let F be a field and let $R = M(n, F)$. Let $x = E_{i,j}$ be the matrix in R which is 1 in the entry in the ith row and jth column, and is 0 in all the remaining $n^2 - 1$ entries. Show that the left ideal $Rx = \{A \in R : A(e_k) = 0 \ \forall \ k \neq j\}$. Show that the right ideal $xR = \{A \in R : A^{tr}(e_k) = 0 \ \forall \ k \neq i\}$.

10. Let F be a field and let $R = F[x]$, the polynomial ring in one variable over F. Let $\alpha, \beta \in F$ and assume that $\alpha \neq \beta$. Let $I_\alpha = (x - \alpha)$ and $I_\beta = (x - \beta)$ be the principal ideals in R generated by $x - \alpha$ and $x - \beta$ respectively. Prove that I_α and I_β are relatively prime.

11. Let R be a commutative ring and suppose that R is a principal ideal ring, i.e., if $I \subset R$ is an ideal, then $I = (a)$ for some $a \in R$. Prove that if $f : R \to S$ is a surjective homomorphism of commutative rings, than S is a principal ideal ring.

12. Let R be a ring with ideals I, J and K. Assume $I + J = R$ and $I + K = R$. Prove that $I + (J \cap K) = R$ and $I + J \cdot K = R$. Prove that if $J_1, \ldots, J_m$ are ideals of R and $I + J_k = R$ for $1 \leq k \leq m$, then $I + (J_1 \cap \cdots \cap J_m) = R$.