SMITH NORMAL FORM OVER THE INTEGERS

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1. INTRODUCTION

In these notes, we explain how to put any \( k \times n \) matrix with integer entries into Smith normal form. The same result is true over an arbitrary principal ideal domain. The same proof works over any Euclidean domain.

2. SMITH NORMAL FORM

Let \( A = (a_{ij}) \) be a \( k \times n \) matrix with entries in the ring \( R = \mathbb{Z} \) of integers.

We say that the matrix \( A \) is in Smith normal form if

1. \( a_{ij} = 0 \) for \( i \neq j \),
2. For some \( m \) with \( 0 \leq m \leq k \), \( a_{ii} \neq 0 \) for \( i \leq m \), and \( a_{ii} = 0 \) for \( i > m \).
3. Let \( a_i = a_{ii} \) for \( 1 \leq i \leq m \). Then \( a_1/a_2/\ldots/a_m \).

If \( m = 0 \) in the above definition, then the matrix \( A \) in Smith normal form is the zero matrix.

Pictorially, a matrix in Smith normal form looks like

\[
A = \begin{bmatrix}
  a_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
  0 & a_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
  0 & 0 & a_3 & 0 & \cdots & \cdots & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \vdots \\
  0 & 0 & \cdots & \cdots & a_m & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}
\]

For a nonzero matrix \( A = (a_{ij}) \) with coefficients in \( R \), we let \( d(A) \) be the greatest positive number dividing all coefficients, so \( d(A) = \gcd\{a_{ij} : 1 \leq i \leq k, 1 \leq j \leq n\} \).
Let $r_i$ be the $i$th row of $A$ and let $c_j$ be the $j$th column of $A$. Recall the row and column operations:

$R_1$: switches the $i$th and $j$th row of $A$.

$R_2$: multiplies the $i$th row of $A$ by $-1$.

$R_3$: replaces the $i$th row of $A$ by $r_i + ar_j$ for some $a \in R$.

$C_1$: switches the $i$th and $j$th columns of $A$.

$C_2$: multiplies the $i$th column of $A$ by $-1$.

$C_3$: replaces the $i$th column of $A$ by $c_i + ac_j$ for some $a \in \mathbb{Z}$.

**Lemma 2.1.** Let $B$ be the result of applying a row or column operation to $A$. Then $d(A) = d(B)$.

**Proof:** For operations $R_1$, $R_2$, $C_1$, and $C_2$ this is trivial. For operations $R_3$ and $C_3$, the claim can be proved in the same way as one proves that $\gcd(a, b) = \gcd(a + sb, b)$ for $a, s, b \in R$.

Q.E.D.

**Proposition 2.2.** Let $A$ be a nonzero matrix with integer coefficients and let $d = d(A)$. Then we can use row and column operations to transform $A$ into a matrix $B$ with $t(B) = d$.

**Proof:** We use induction on the minimum element $t = t(A)$ of the set $\{|a_{ij}| : a_{ij} \neq 0, 1 \leq i \leq k, 1 \leq j \leq n\}$. Clearly, $t \geq d$. If $t = d$, we're done. If not, we may assume the result is known for matrices $C$ with integer coefficients such that $d(C) = d$ and $t(C) < t$.

Let $|a_{uv}| = t$, and note that by an operation of type $R_2$ by $-1$, we can assume that $a_{uv} = t$. Since $t > d$, there exists a coefficient $a_{lm}$ of $A$ such that $t$ does not divide $a_{lm}$ (otherwise, $t$ would divide $d$). We have to consider two cases:

**Case 1:** There exists an entry $a_{lm}$ such that $t$ does not divide $a_{lm}$ with $a_{lm}$ in either the same column or row as $a_{uv}$ (i.e., $u = l$ or $v = m$). By the Euclidean algorithm, we can write $a_{lm} = qa_{uv} + r$ for some integer $r$ such that $0 < r < a_{uv}$. If $v = m$, so $a_{uv}$ and $a_{lm} = a_{iv}$ are in the same column, then by replacing the $l$th row of $A$ by $r_i - qr_u$ (lth row - q times uth row), we get a new matrix $B$. By Lemma 2.1, $d(A) = d(B)$. By construction the $lm$ entry $b_{lm}$ of $B$ is $r$. Hence, $t(B) = r < t$, so by the inductive assumption, we can perform row and column operations on $B$ to transform $B$ to a new matrix $C$ with $t(C) = d$.

**Case 2:** $a_{uv}$ divides every entry $a_{uj}$ and $a_{iv}$ in the same row or column as $a_{uv}$. Then note that we can do row and column operations to transform $A$ so that the $uv$ row and $uv$ column are 0 outside of the $a_{uv}$-entry. Indeed, we begin with the entries of the form $a_{iv}$. Since $a_{uv}$ divides $a_{iv}$ by assumption, there exists $z \in R$ such that $a_{iv} = za_{uv}$. Then a row operation replacing the row $r_i$ by $r_i - zr_u$ has the effect of making the $a_{iv}$ entry equal to 0. None of these operations change the $uv$ row. Now each entry $a_{uj} = za_{uv}$ for some $z \in R$, and a column operation replacing the $j$th column by $c_j - zc_v$ makes the $a_{uj}$ entry
equal to 0. Call this new matrix $B$. Note that $d(B) = d$ by Lemma 2.1, and $t(B) \leq t$ since $b_{uv} = a_{uv} = t$. If $t(B) = d$, we're done. If not, $t(B) > d$, and it follows that there is an entry $b_{lm}$ such that $b_{uv}$ does not divide $b_{lm}$. By construction, $u \neq l$ and $v \neq m$. We replace the $uth$ row of $B$ by $r_l + r_u$ and call this new matrix $C$. Since $b_{lv} = 0$ by construction, this does not change the entry $b_{uv}$, so $t(C) \leq t$. However, it makes the entry $c_{um} = b_{um} + b_{lm} = 0 + b_{lm} = b_{lm}$, since $b_{um} = 0$ by construction. Now we reason as in case 1, to show that we can change $C$ by a column operation to get a new matrix $D$ so that $t(D) < t(C) \leq t$, but $d(D) = d$. Hence, by induction, we can transform $D$ by column operations to a new matrix $X$ with $t(X) = d$.

Q.E.D.

**Theorem 2.3.** If $A$ is a matrix with integer coefficients, we can transform $A$ into Smith normal form using row and column operations.

**Proof:** If $A = 0$, the result is trivial. If $A \neq 0$, then we can use Proposition 2.2 to transform $A$ by row and column operations into a matrix with entry $a_{uv}$ with $a_{uv} = d = d(A)$. By definition, this means that $a_{uv}$ divides all entries of $A$. By using operations of type $R_1$ and $C_1$, transform the matrix $A$ into a new matrix $A$ with $a_{11} = d$. Perform row and column operations of type $R_3$ and $C_3$ to zero out the first row and column. Note that since $d$ divides all entries of $A$, then $d$ divides all entries of this new matrix, still denoted $A$. The new matrix has the form

$$A = \begin{bmatrix} d & 0 \\ 0 & B \end{bmatrix},$$

where $B$ is a $k - 1$ by $n - 1$ matrix with integer entries, all divisible by $d$, and the 2 0’s denote zero vectors of the appropriate size. Let $a_1 = d$, and perform row and column operations to the matrix $B$ (not involving the first row or column) to put $B$ into similar form. The remainder of the theorem follows by an easy induction.

Q.E.D.

The proof gives a procedure for reducing a matrix into Smith normal form. One can compute $t(A)$ and $d(A)$. If $d(A) < t(A)$, then one can use the argument from the proof of the Proposition to transform $A$ into a new matrix $B$ with $t(B) < t(A)$. Keep doing this until we get to a matrix $C$ with $t(C) = d(C) = d(A)$. Then it’s easy to zero out all entries in the first row and column aside from the $a_{11}$.

I suspect there are much more efficient algorithms, but anyway this argument gives a proof of our result on Smith normal form.