NOTES ON THE ZARISKI TANGENT SPACE

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Let X be an affine algebraic set. We sketch definitions and basic properties of vector fields and tangent spaces at points of X.

For perspective, recall how we define the tangent space of a differentiable manifold M. We cover M by open neighborhoods U_i which are identified with $\mathbb{R}^n$, and then we transfer our understanding of the tangent space at a point of $\mathbb{R}^n$ to define the tangent space at a point in U_i. This can be shown to be independent of choices. This approach is not a good idea for an affine algebraic set X because X does not have an open cover by Zariski open sets that are identified with an open set in some $\mathbb{C}^n$. First of all, if this were the case, then X would be smooth, so we would miss information about singularities, but secondly even smooth affine varieties are not necessarily locally isomorphic to some $\mathbb{C}^n$.

We first discuss vector fields.

Remark 0.1. *Let R be a finitely generated $\mathbb{C}$-algebra and let M be a R-module. A M-valued derivation of R is a $\mathbb{C}$-linear map $D : R \to M$ such that the Leibniz rule, $D(fg) = fD(g) + gD(f)$ for all $f, g \in R$, is satisfied.*

Note that it follows from definitions that $D(a) = 0$ if $a \in \mathbb{C}$ is a constant.

Remark 0.2. *Let X be an affine set. By definition, a vector field on X is a $\mathbb{C}[X]$-valued derivation of $\mathbb{C}[X]$.*

We compute vector fields on $X = \mathbb{C}^n$. Let $R = \mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$. Let $D : R \to R$ be a R-valued derivation. Then it follows from the Leibniz rule that D is determined by $D(x_i)$. Consider the operator $\xi_D : R \to R$ given by $\xi_D = \sum_{i=1}^n D(x_i)\partial_i$, where ∂_i is partial differentiation with respect to x_i. It is easy to check that $\xi_D = D$. In particular, every derivation is an expression of the form $\sum a_i\partial_i$ with $a_i \in R$, so vector fields on X are just ordinary vector fields with polynomial coefficients.

Note that a vector field on X is determined by its values on the generators of $\mathbb{C}[X]$.

Remark 0.3. *EXERCISE 1 Compute the vector fields on $V(y^2 - x^3)$. Show that they are all of the form $\xi_{a,b} = a\partial_x + b\partial_y$ where $a, b \in \mathbb{C}[x, y]/(y^2 - x^3)$. What conditions must a and b satisfy for $\xi_{a,b}$ to be a vector field. Compute the vector fields on $V(y - x^3)$. Show that they are the same as vector fields on $\mathbb{C}$.*

In ordinary calculus, we can obtain tangent vectors at a point in $\mathbb{R}^n$ by specializing vector fields at that point. For example, the vector field $e^x\partial_x + \sin(y)\partial_y$ specializes to the tangent vector ∂_x by evaluating at $(0,0)$. We can do the same thing with our definition of vector fields as follows.

Let $R = \mathbb{C}[X]$ and let $D : R \to R$ be a R-valued derivation and let $\alpha \in X$ be a point. Identify $R/\mathfrak{m}_\alpha \cong \mathbb{C}$ via $f + \mathfrak{m}_\alpha \mapsto f(\alpha)$. Then we obtain a $\mathbb{C}$-valued derivation $D_\alpha : R \to \mathbb{C}$ by the formula $D_\alpha(f) = (D(f))(\alpha)$. Then it is easy to check that D_α is a $\mathbb{C}$-valued derivation and if $f, g \in R$, then $D_\alpha(fg) = f(\alpha)D_\alpha(g) + g(\alpha)D_\alpha(f)$. Since in ordinary calculus, all tangent vectors arise by specialization of vector fields, it is somewhat natural to define the Zariski tangent space as follows.

Remark 0.4. *If $\alpha \in X$, then the Zariski tangent space $T_\alpha(X)$ to X at α is the set of all $\mathbb{C}$-valued derivations D of R such that $D(fg) = f(\alpha)D(g) + g(\alpha)D(f)$ for all $f, g \in R$. A $\mathbb{C}$-valued derivation of R as above is then called a tangent vector at α.*

It is easy to see that $T_\alpha(X)$ is a complex vector space under addition of derivations. It is also not difficult to show that a tangent vector D is determined by its value on generators of $\mathbb{C}[X]$. Given this, the reader can easily show that $T_\alpha(\mathbb{C}^n)$ is the $\mathbb{C}$-span of $\partial_{j,\alpha}$, where $\partial_{j,\alpha}(f) = (\partial_j(f))(\alpha)$.

We use this to identify $T_\alpha(\mathbb{C}^n) \cong \mathbb{C}^n$, by letting $(b_1, \ldots, b_n) \in \mathbb{C}^n$ correspond to $b_1\partial_{1,\alpha} + \cdots + b_n \partial n, \alpha$.

There is an alternative definition of the Zariski tangent space at a point α which emphasizes more the role of maximal ideals. Let $\mathfrak{m} = \mathfrak{m}_\alpha$ be the maximal ideal of α. Then $\mathfrak{m}/\mathfrak{m}^2$ is naturally a R-module with trivial $\mathfrak{m}$-action and hence is a $\mathbb{C} \cong R/\mathfrak{m}$ vector space. Since R is Noetherian, $\mathfrak{m}$ is a finitely generated R-module, and hence its quotient $\mathfrak{m}/\mathfrak{m}^2$ is a finitely generated R-module with trivial $\mathfrak{m}$-action. Thus, $\mathfrak{m}/\mathfrak{m}^2$ is a finitely generated $R/\mathfrak{m} = \mathbb{C}$-module, so $\mathfrak{m}/\mathfrak{m}^2$ is a finite dimensional complex vector space.

Remark 0.5. *The Zariski cotangent space $T^*_\alpha(X)$ is the finite dimensional $\mathbb{C}$-vector space $\mathfrak{m}/\mathfrak{m}^2$.*

For $f \in R$, let $df := df(\alpha) := f - f(\alpha) + (\mathfrak{m}^2) \in \mathfrak{m}/\mathfrak{m}^2$. It follows from an easy calculation that $d(fg)(\alpha) = f(\alpha)dg(\alpha) + g(\alpha)df(\alpha)$. Further, if $X = \mathbb{C}^n$, then $df = \sum_{i=1}^n \partial_i(f)(\alpha)dx_i$. This is the usual formula for the de Rham differential in calculus, and it can be checked easily for monomials and follows in general by linearity.

Further, note that $dx_i(\partial_j) = \partial_j(x_i + \mathfrak{m}^2)(\alpha) = \delta_{ij}$.

We would like the cotangent space to be the linear dual of the tangent space. This follows from the following result.

Proposition 0.6. *The linear dual $(\mathfrak{m}/\mathfrak{m}^2)^* \cong T_\alpha(X)$. In particular, $T_\alpha(X)$ is a finite dimensional vector space.*

Proof : To prove this, identify $\mathbb{C}$ with constant functions on X. Then $R = \mathbb{C}[X] = \mathbb{C} \oplus \mathfrak{m}$ as vector spaces. Define a map $\chi : T_\alpha(X) \to (\mathfrak{m}/\mathfrak{m}^2)^*$ by $\chi(D)(f) = D(f)$. To check $\chi(D)$ is well-defined, note that if $f, g \in \mathfrak{m}$, then $\chi_D(f \cdot g) = f(\alpha)D(g) + g(\alpha)D(f) = 0$ since $f, g \in \mathfrak{m} = \mathfrak{m}_\alpha$. It follows from definitions that $\chi_D(\mathfrak{m}^2) = 0$. Conversely, if $\eta \in (\mathfrak{m}/\mathfrak{m}^2)^*$, we may regard η as a linear map $\eta : \mathfrak{m} \to \mathbb{C}$ such that $\eta(\mathfrak{m}^2) = 0$. Define $D_\eta : R \to \mathbb{C}$ by setting $D_\eta(c) = 0$ if $c \in \mathbb{C}$ and $D_\eta(f) = \eta(f)$ for $f \in \mathfrak{m}$. This defines D_η uniquely, and the reader can check that D_η is a derivation. It is not difficult to check that χ and $\eta \mapsto D_\eta$ are inverses, which completes the proof of the Proposition.

Q.E.D.

Using this result, we can show that the tangent space can be computed using a neighborhood of a point.

Lemma 0.7. *Let X be an affine algebraic set and let $U \subset X$ be a principal open set and consider a point $\alpha \in U$. Then $T_\alpha(U) \cong T_\alpha(X)$.*

Proof : Let $\mathfrak{m} = \mathfrak{m}_\alpha \in \mathbb{C}[X]$ be the maximal ideal of functions on X vanishing at α. Let $U = X_f$ and let $S = (f^k)$. Then $\mathfrak{n} := S^{-1}\mathfrak{m}$ is the maximal ideal of functions in $\mathbb{C}[U]$ vanishing at α. By Proposition 0.6, it suffices to prove that $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{n}/\mathfrak{n}^2$. For this, verify the following easy fact: let R be a ring and let M be a R-module, and let $S \subset R$ be a multiplicative set such that for all $s \in S$, the map $l_s : M \to M, x \mapsto s \cdot x$ is a R-module isomorphism. Then it follows from definitions that the canonical map $M \to S^{-1}M$ given by $x \mapsto \frac{x}{1}$ is an isomorphism of R-modules. Since $\alpha \in X_f$, $f(\alpha) \neq 0$, so $f \notin \mathfrak{m}$. It follows that $f + \mathfrak{m}$ is nonzero in the field $A/\mathfrak{m}$, and l_f acts as an isomorphism on the A-module $\mathfrak{m}/\mathfrak{m}^2$, since A acts on $\mathfrak{m}/\mathfrak{m}^2$ through its quotient $A/\mathfrak{m}$. Thus the above easy fact applies to give $\mathfrak{m}/\mathfrak{m}^2 \cong S^{-1}(\mathfrak{m}/\mathfrak{m}^2)$. But $S^{-1}(\mathfrak{m}/\mathfrak{m}^2) \cong (S^{-1}\mathfrak{m})/(S^{-1}\mathfrak{m})^2$ by exactness of localization. Hence $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{n}/\mathfrak{n}^2$, which completes the proof.

Q.E.D.

It is useful to define the tangent space for an affine algebraic set $X = V(I) \subset \mathbb{C}^n$. As a set,

$TX = \{(\alpha, D) : \alpha \in X, D \in T_\alpha(X)\}$, and we would like to give TX the structure of an affine algebraic set. For this, it is convenient to introduce an auxiliary ring.

Remark 0.8. *The ring of dual numbers is the ring $\mathbb{C}[t]/(t^2)$. We let $\delta = t + (t^2)$, so the ring of dual numbers is*

$\mathbb{C}[\delta] = \{a + b\delta : a, b \in \mathbb{C}, \delta^2 = 0\}$.

Define $p : \mathbb{C}[\delta] \to \mathbb{C}$ by $p(a + b\delta) = a$. It is routine to check that p is an algebra homomorphism.

Lemma 0.9. *Let $R = \mathbb{C}[X]$ be the ring of functions on an affine algebraic set $X \subset \mathbb{C}^n$ and let $I = I(X)$. There is a bijection $\eta : \mathrm{Hom}_{alg}(R, \mathbb{C}[\delta]) \to T(X)$ between the collection of algebra homomorphisms from R to the ring of dual numbers and the tangent space.*

Proof : If $\phi : R \to \mathbb{C}[\delta]$ is an algebra homomorphism, let $\phi(f) = a(f) + b(f)\delta$. Then $f \mapsto p_1 \circ \phi(f) = a(f)$ is an algebra homomorphism, so there is a maximal ideal $\mathfrak{m}_\alpha$ with $\alpha \in X$ such that $\mathfrak{m}_\alpha$ is the kernel of $f \mapsto a(f)$. The reader can check that $f \mapsto D(f) :=$

$b(f)$ is a derivation and we define η by the formula $\eta(\phi) := (\alpha, D) \in T(X)$. Conversely, if $(\alpha, D) \in T(X)$, define $\phi_{\alpha,D} : R \to \mathbb{C}[\delta]$ by the formula $\phi_{\alpha,D}(f) = f(\alpha) + D(f)\delta$. It is routine to check that ϕ is an algebra homomorphism, and that $\phi \mapsto \eta(\phi)$ and $(\alpha, D) \mapsto \phi_{\alpha,D}$ are inverse equivalences. This completes the proof of the Lemma.

Q.E.D.

Let $A = \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[\mathbb{C}^n]$. Let $\phi : A \to \mathbb{C}[\delta]$ be an algebra homomorphism. Then $\phi(x_i) = a_i(\phi) + b_i(\phi)\delta$ for some $a_i(\phi), b_i(\phi) \in \mathbb{C}$. Since ϕ is determined by its value on generators, it is routine to check that the map

$\psi : \mathrm{Hom}_{alg}(A, \mathbb{C}[\delta]) \to T(\mathbb{C}^n) = \mathbb{C}^n \times \mathbb{C}^n$, $\psi(\phi) = (a_1(\phi), \ldots, a_n(\phi); b_1(\phi), \ldots, b_n(\phi))$ is bijective. We treat this bijection as an identification, and use it to regard $\mathrm{Hom}_{alg}(A, \mathbb{C}[\delta])$ as the algebraic variety $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$.

Proposition 0.10. *Let X be an affine algebraic set and let $R = \mathbb{C}[X] = A/I$ as above. Use Lemma 0.9 to identify $T(X) = \mathrm{Hom}_{alg}(R, \mathbb{C}[\delta])$. Identify $\mathrm{Hom}_{alg}(R, \mathbb{C}[\delta]) = \{\phi \in T(\mathbb{C}^n) : \phi(I) = 0\}$. Then under these identifications, $T(X)$ is a closed algebraic set in $T(\mathbb{C}^n) = \mathbb{C}^n \times \mathbb{C}^n$. Further, the map $p : T(X) \to X$ given by $p((\alpha, D)) = \alpha$ and the map $i : X \to T(X)$ given by $i(\alpha) = (\alpha, 0)$ are both morphisms.*

Proof : To prove this, note that $\mathrm{Hom}_{alg}(R, \mathbb{C}[\delta]) = \{\phi \in \mathrm{Hom}_{alg}(A, \mathbb{C}[\delta]) : \phi(I) = 0\}$, by the universal property of quotient rings. Further, note that if $\phi \in \mathrm{Hom}_{alg}(A, \mathbb{C}[\delta])$, and $I = (f_1, \ldots, f_r)$, then $\phi(I) = 0$ if and only if $\phi(f_1) = \cdots = \phi(f_r) = 0$. Thus, if $\phi \in \mathrm{Hom}_{alg}(A, \mathbb{C}[\delta])$, $\phi \in T(X)$ if and only if $\phi(f_j) = 0$ for all $j = 1, \ldots, r$.

We have identified $\phi \in \mathrm{Hom}_{alg}(A, \mathbb{C}[\delta])$ as a point of $\mathbb{C}^n \times \mathbb{C}^n$ via the map

$\phi \mapsto (a_1, \ldots, a_n; b_1, \ldots, b_n)$, where $\phi(x_i) = a_i + b_i\delta$. Let $f_j = \sum c_E {x_1}^{e_1} \ldots {x_n}^{e_n}$, where $E = (e_1, \ldots, e_n)$ runs through collections of nonnegative integers. Then

$\phi(f_j) = \sum c_E (a_1 + b_1\delta)^{e_1} \ldots (a_n + b_n\delta)^{e_n} = r_j + s_j\delta$ for some $r_j, s_j \in \mathbb{C}$. We compute r_j and s_j by using the formula $(a + b\delta)^k = a^k + ka^{k-1}b\delta$. It follows that $\phi(f_j) = 0$ if and only if $r_j = s_j = 0$.

The constant coefficient r_j is:

A(j): $\sum_E c_E {a_1}^{e_1} \ldots {a_n}^{e_n} = 0$, and $r_j = 0$ if and only if the point $(a_1, \ldots, a_n) \in V(f_j)$.

The δ coefficient s_j is:

B(j): $\sum_E c_E \sum_{i=1}^n {a_1}^{e_1} \ldots \eta({a_i}^{e_i}){a_n}^{e_n}$, where $\eta({a_i}^{e_i}) := e_i {a_i}^{e_i - 1}$.

Since a point $(a_1, \ldots, a_n; b_1, \ldots, b_n)$ attached to a homomorphism $\phi : A \to \mathbb{C}[\delta]$ corresponds to a tangent vector in $T(X)$ if and only if $\phi(f_j) = 0$ for all $j = 1, \ldots, r$, it follows that we may identify

$T(X)$ with the set of points $(a_1, \ldots, a_n; b_1, \ldots, b_n) \in \mathbb{C}^n \times \mathbb{C}^n$ where the polynomial identities $A(j)$ and $B(j)$ are satisfied for $j = 1, \ldots, r$. In particular, $T(X)$ may be identified with an affine algebraic subset of $\mathbb{C}^n \times \mathbb{C}^n$, which establishes the first part of the proposition.

For the remainder, note that the map $\tilde{p} : T(\mathbb{C}^n) = \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ given by $\tilde{p}(\alpha, D) = \alpha$ is a morphism, and p is the restriction of $\tilde{p}$ to $T(X)$, so p is a morphism. A similar argument shows that i is a morphism.

Q.E.D.

Corollary 0.11. *(OF LAST PROOF) Let $X = V(I) \subset \mathbb{C}^n$ where $I = (f_1, \ldots, f_r)$. Then*

$$T_\alpha(X) = \{D = \sum_{j=1}^{n} b_j \partial_j \in T_\alpha(\mathbb{C}^n) : df_k(D) = \sum_{j=1}^{n} \partial_j(f_k)(\alpha) \cdot b_j = 0, k = 1, \ldots, r\}.$$

Proof : Indeed, $D \in T_\alpha(X)$ if and only if $(\alpha, D) \in T(X)$, which is true if and only if D satisfies the equations $B(k)$ for $k = 1, \ldots, r$ with $\alpha = (a_1, \ldots, a_n)$. It is easy to check that the equation $B(k)$ is equivalent to the condition that $df_k(D) = 0$.

Q.E.D.

It is useful to think about these result in a simple example. Let $X = V(y^2 - x^3)$, where $x = x_1$ and $y = x_2$ as usual. The Proposition identifies $T(X)$ with the set of points $\{(a_1, a_2; b_1, b_2)\}$ satisfying the identities
${a_2}^2 = {a_1}^3$, and $2a_2b_2 = 3{a_1}^2 b_1$.

Remark 0.12. *Even if X is an affine variety, $T(X)$ need not be an affine variety.*

Remark 0.13. *EXERCISE T Let $X = V(y^2 - x^3) \subset \mathbb{C}^2$. Show that $T(X)$ is not an irreducible affine algebraic subset of $\mathbb{C}^4$. In particular, show $T_{(0,0)}(X)$ is an irreducible component of $T(X)$. Find another irreducible component of $T(X)$.*

Remark 0.14. *(EXERCISE 2) Compute the tangent space to $V(x^2 - yz)$ at $(0, 0, 0)$ in $\mathbb{C}^3$. What is its dimension? Compute the tangent space at any point $\alpha \in V(x^2 - yz)$ besides $(0, 0, 0)$. What is its dimension?*

Remark 0.15. *Let X be an affine variety and let $\alpha \in X$. We will show that $\dim(T_\alpha(X)) \geq \dim(X)$. α is called a smooth point of X if $\dim(T_\alpha(X)) = \dim(X)$, and α is called a singular point of X if $\dim(T_\alpha(X)) > \dim(X)$. An affine variety X is called smooth or nonsingular if all of its points are smooth, and otherwise is called singular. If X is an affine algebraic set, we may say X is smooth if its irreducible components are its connected components and each irreducible component is smooth. Otherwise, we say X is singular. The intuition is that on the affine algebraic set $V(xy)$ which is one-dimensional, the tangent space at $(0, 0)$ is two dimensional since ∂_x and ∂_y evaluate at $(0, 0)$ to give linearly independent derivations. Thus, the Zariski tangent space is bigger than we would expect for a smooth variety of dimension 1, so $V(xy)$ is singular at $(0, 0)$. In general, if X_1, X_2 are two irreducible components of an affine algebraic set X which meet at a point α, then X should be singular at α since there are tangent vectors tangent to X_2 but not to X_1 and vice versa, so the Zariski tangent space is too big.*

Remark 0.16. *Let $U \subset X$ be an affine open subset of an affine variety X and let $\alpha \in U$. Then by Lemma 0.7, α is a smooth point of U if and only if α is a smooth point of X.*

Remark 0.17. *Let X be an affine variety and let S be the set of singular points of X, and denote by $X_r := X - S$. We will prove that S is a closed subset of X and X_r is an open, dense subset of X. To prove these assertions, we will proceed as follows:*

(1) Prove the assertions for an irreducible hypersurface of affine space $\mathbb{C}^n$.

(2) Use upper semi-continuity of dimension to prove that S is closed.

(3) Let $\dim(X) = d$. We show there exists a birational morphism $\phi : X \to V(f)$, where $V(f) \subset \mathbb{C}^{d+1}$ is the zero set of an irreducible polynomial.

(4) Show that a morphism of varieties induces a morphism of tangent spaces with good properties.

(5) Use invariance properties of tangent space under an isomorphism and (3) to show that the special case (1) implies the assertion in general.

We begin with step (1).

Proposition 0.18. *Let $f \in \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[\mathbb{C}^n]$ be irreducible and let $Y = V(f)$ be the corresponding affine variety. Then the singular set S of Y is a proper closed subset.*

Proof : Since f is irreducible, it follows easily that (f) is a prime ideal of $\mathbb{C}[x_1, \ldots, x_n]$, so Y is irreducible. Further, $\dim(Y) = n - 1$ by Theorem 0.8 of the notes on fiber dimension. By Corollary 0.11 of these notes, if $\alpha \in Y$, then

$$T_\alpha(Y) = \{\sum a_i \partial_i \in T_\alpha(\mathbb{C}^n) : \sum a_i \partial_i(f)(\alpha) = 0\}.$$

Then either the vector $df(\alpha) := \partial_1(f)(\alpha)dx_1 + \ldots \partial_n(f)(\alpha))dx_n = 0$ or it is nonzero, and the tangent space $T_\alpha(Y)$ is the subspace annihilated by $df(\alpha)$. If it is zero, then $T_\alpha(Y)$ is n-dimensional so α is a singular point. If it is nonzero, then $T_\alpha(Y)$ is $n - 1$-dimensional, so α is a smooth point. It follows that $S = \cap_{i=1}^n V(\partial_i(f))$ is closed.

If every $\alpha \in Y$ were a singular point, then $\partial_i(f)(\alpha) = 0$ for all $\alpha \in Y$ and $i = 1, \ldots, n$, so by the Nullstellensatz, $\partial_i(f) \in I(Y) = (f)$. But the degree of $\partial_i(f)$ is less than the degree of f (by convention, degree of zero polynomial is -1), so if f divides $\partial_i(f)$, it follows that $\partial_i(f) = 0$ for all i, so f is constant. This contradicts the assumption that f is irreducible, and completes the proof.

Q.E.D.

We now establish a generalization of Step (2).

Proposition 0.19. *Let X be an affine variety. Let $S_k(X) := \{\alpha \in X : \dim(T_\alpha(X)) \geq k\}$. Then $S_k(X)$ is closed in X.*

Proof : Let $TX = \cup_{i \in I} T_i$ be the decomposition of the affine algebraic set TX into irreducible components. Let $p_i : T_i \to X$ be the restriction of $p : TX \to X$ to T_i. For

$\alpha \in X$, let $S_k(p_i)$ be the set of $(\alpha, D) \in T_i$ such that there is an irreducible component of $p_i^{-1}(\alpha)$ of dimension at least k. $S_k(p_i)$ is the subset of T_i associated to the morphism $p_i : T_i \to X$ in Theorem 0.24 of the notes on fiber dimension.

If $\alpha \in X$, then $T_\alpha(X) = p^{-1}(\alpha) = \cup_{i \in I} p_i^{-1}(\alpha)$ is a vector space. Thus, $T_\alpha(X)$ is irreducible, so since each $p_i^{-1}(\alpha)$ is closed, $T_\alpha(X) = p_i^{-1}(\alpha)$ for some $i \in I$. In particular, (*) There is $i \in I$ such that $p_i^{-1}(\alpha)$ is irreducible, and $\dim(T_\alpha(X))$ is the maximum dimension of the irreducible components of $p_i^{-1}(\alpha)$ among $i \in I$.

Let $S_k(p) = \cup_{i \in I} S_k(p_i)$. By (*), $S_k(p) = \{(\alpha, D) : \dim(T_\alpha(X)) \geq k\}$.

Since $S_k(p_i)$ is closed for each $i \in I$ by Theorem 0.24 of the notes on fiber dimension, it follows that $S_k(p)$ is closed. Since $(\alpha, 0) \in T_\alpha(X)$ for all $\alpha \in X$, it follows that $S_k(X) = i^{-1}(S_k(p))$, and hence $S_k(X)$ is closed in X.

Q.E.D.

Proposition 0.20. *Let $Sing(X)$ be the set of singular points of an affine variety X. Then $Sing(X)$ is closed.*

Proof : By definition, $Sing(X) = S_{d+1}(X)$, where $d = \dim(X)$. Now apply Proposition 0.19.

Q.E.D.

For Step (3), we want to prove that every affine variety is birational to a hypersurface.

For this, we recall some results about unique factorization domains (UFD's) (see Dummit and Foote, 9.3, or Ash, section 2.9). This is standard material, but I didn't find it explained in the literature in the needed form.

Let R be a unique factorization domain and let $F = Frac(R)$ be its fraction field. Let $m(x) \in F[x]$. We can write $m(x) = a_n x^n + \cdots + a_0$ with $a_i = \frac{b_i}{c_i} \in F$, where $b_i, c_i \in R$ and $a_n \neq 0$. Let $d = c_n \cdots c_0$ be the product of the denominators of the coefficieents, so $d \cdot m(x) = m_1(x) \in R[x]$. For a polynomial $f(x) \in R[x]$, let $c = c(f)$ be the greatest common divisor of its coefficients. c is called the content of f. Then $m_1(x) = c \cdot m_0(x)$ with $m_0(x) \in R[x]$. Then $m_0(x)$ is primitive, i.e., the greatest common divisor of its coefficients is 1. Thus, $m(x) = \frac{c}{d} m_0(x)$ with $m_0(x) \in R[x]$.

Proposition 0.21. *(Dummit and Foote, Corollary 6 of 9.3) Let $f(x) \in R[x]$ be irreducible in $F[x]$ and also primitive. Then $f(x)$ is irreducible in $R[x]$.*

It follows from definitions that if the polynomial $m(x)$ is irreducible in $F[x]$, then $m_0(x)$ is irreducible in $F[x]$, so by the Proposition, $m_0(x)$ is irreducible in $R[x]$.

Lemma 0.22. *Let R be a UFD with fraction field F and let $m(x) \in F[x]$ be irreducible. Then there exists an irreducible polynomial $m_0(x) \in R[x]$ such that $R[x]/(m_0(x))$ is an integral domain with fraction field $F[x]/(m(x)) = F[x]/(m_0(x))$.*

Proof : To prove this, construct $m_0(x)$ from $m(x)$ as above, and note that $m(x)$ and $m_0(x)$ generate the same ideal in $F[x]$. Consider the ring homomorphism $R[x]/(m_0(x)) \to F[x]/F[x] \cdot m_0(x)$ induced by the obvious ring homomorphism $R[x] \to F[x]$. Suppose $p \in R[x]$ and assume $p \in F[x] \cdot m_0(x)$. Then $p = h \cdot m_0(x)$ for some $h \in F[x]$. By Gauss's Lemma (see Proposition 5 of 9.3 in [DF]), it follows that $m_0(x)$ divides p in $R[x]$. Hence $p + (m_0(x)) = 0$ in $R[x]/(m_0(x))$, so the ring homomorphism $R[x]/(m_0(x)) \to F[x]/(m_0(x))$ is injective. It follows that $R[x]/(m_0(x))$ is an integral domain, and it is easy to show its fraction field is $F[x]/(m_0(x))$.

Q.E.D.

Remark 0.23. *Let $\eta : R \to S$ be an injective homomorphism of integral domains. There is an induced field homomorphism $\tilde{\eta} : Frac(R) \to Frac(S)$ with the property that $\tilde{\eta}(\frac{a}{b}) = \frac{\eta(a)}{\eta(b)}$. We call $\tilde{\eta}$ the localization of η.*

Lemma 0.24. *Let X and Y be affine varieties and suppose there exists a $\mathbb{C}$-algebra isomorphism of the function fields $\chi : \mathbb{C}(Y) \to \mathbb{C}(X)$. Then there exists $c \in \mathbb{C}[X]$ and a birational morphism $\phi : X_c \to Y$ such that $\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X_c]$ extends under localization to $\chi : \mathbb{C}(Y) \to \mathbb{C}(X) = Frac(\mathbb{C}[X_c])$.*

Proof : Since $\mathbb{C}[Y]$ and $\mathbb{C}[X]$ are integral domains, they are subrings of their fraction fields. Let $\mathbb{C}[Y] = \mathbb{C}[\alpha_1, \dots, \alpha_n]$ be generated by $\alpha_1, \dots, \alpha_n$, so $\chi(\mathbb{C}[Y]) = \mathbb{C}[\chi(\alpha_1), \dots, \chi(\alpha_n)]$. Let $\chi(\alpha_i) = \frac{b_i}{c_i}$, with $b_i, c_i \in \mathbb{C}[X]$. Let $c = c_1 \cdots c_n$. Thus, $\chi(\mathbb{C}[Y]) \subset \mathbb{C}[X]_c$, so χ restricts to give an injective ring homomorphism $\chi : \mathbb{C}[Y] \to \mathbb{C}[X_c]$, and hence a morphism of varieties $\phi : X_c \to Y$ such that $\chi = \phi^*$. Then ϕ is dominant since χ is injective, and the induced field homomorphism $\chi : \mathbb{C}(Y) \to \mathbb{C}(X)$ is an isomorphism by assumption, so $\phi : X_c \to Y$ is birational.

Q.E.D.

Remark 0.25. *EXERCISE Let $X = V(y^2 - x^3)$ and let $Y = \mathbb{C}$. Give a field isomorphism $\chi : \mathbb{C}(Y) \to \mathbb{C}(X)$ and find $c \in \mathbb{C}[X]$ and the morphism $\phi : X_c \to Y$ inducing χ.*

Proposition 0.26. *Let X be a d-dimensional affine variety. Then there exists nonzero $c \in \mathbb{C}[X]$, an irreducible polynomial $f \in \mathbb{C}[\mathbb{C}^{d+1}]$ and a birational morphism $\phi : X_c \to V(f)$.*

Proof : By the Noether normalization Lemma, there exist $\beta_1, \dots, \beta_d \in \mathbb{C}[X]$ that are algebraically independent over $\mathbb{C}$ such that $\mathbb{C}[X]$ is integral over $\mathbb{C}[\beta_1, \dots, \beta_d]$. It follows that $\mathbb{C}(X)$ is algebraic over $\mathbb{C}(\beta_1, \dots, \beta_d)$. Since these fields have characteristic zero, this field extension is separable, so by the theorem of the primitive element (Dummit and Foote, Theorem 25 of 14.4), $\mathbb{C}(X) = \mathbb{C}(\beta_1, \dots, \beta_d)[\alpha]$, where $\alpha \in \mathbb{C}(X)$ is an element algebraic over $\mathbb{C}(\beta_1, \dots, \beta_d)$. Let $m(y)$ be the minimal polynomial of α over $\mathbb{C}(\beta_1, \dots, \beta_d)$, so that $\mathbb{C}(\beta_1, \dots, \beta_d)[\alpha] \cong \mathbb{C}(\beta_1, \dots, \beta_d)[y]/(m(y))$. Note that $\mathbb{C}[\beta_1, \dots, \beta_d] \cong \mathbb{C}[x_1, \dots, x_d]$ is a UFD since the collection $\beta_1, \dots, \beta_d$ is algebraically independent. Since $m(y)$ is irreducible over $\mathbb{C}(\beta_1, \dots, \beta_d)$ by Lemma 0.22, there exists $p(y) \in \mathbb{C}[\beta_1, \dots, \beta_d][y]$ with $p(\alpha) = 0$, $p(y)$

irreducible and such that $\mathbb{C}[\beta_1,\dots,\beta_d][y]/(p(y))$ injects into $\mathbb{C}(\beta_1,\dots,\beta_d)[y]/(p(y)) \cong \mathbb{C}(\beta_1,\dots,\beta_d)[\alpha] = \mathbb{C}(X)$.

Consider the unique $\mathbb{C}$-algebra isomorphism $\psi : \mathbb{C}[x_1,\dots,x_d] \to \mathbb{C}[\beta_1,\dots,\beta_d]$ such that $\psi(x_i) = \beta_i$, which extends to a $\mathbb{C}$-algebra isomorphism $\mathbb{C}[x_1,\dots,x_d][y] \to \mathbb{C}[\beta_1,\dots,\beta_d][y]$ mapping y to y. Let $\psi(q(y)) = p(y)$. This induces a $\mathbb{C}$-algebra isomorphism $\psi_1 : \mathbb{C}[x_1,\dots,x_d][y]/(q(y)) \to \mathbb{C}[\beta_1,\dots,\beta_d][y]/(p(y))$. The localization χ of ψ_1 is a field isomorphism $\chi : \mathbb{C}(x_1,\dots,x_d)[y]/(q(y)) \to \mathbb{C}(\beta_1,\dots,\beta_d)[y]/(p(y)) \cong \mathbb{C}(X)$.

Let $Y = V(q) \subset \mathbb{C}^{d+1}$. Since $p(y)$ is irreducible, $q(y)$ is irreducible, and it follows that Y is an irreducible hypersurface in $\mathbb{C}^{d+1}$, where y is regarded as the $d+1$th variable. Further, $\mathbb{C}(Y) = \mathbb{C}(x_1,\dots,x_d)[y]/(q(y))$, so $\chi : \mathbb{C}(Y) \to \mathbb{C}(X)$ is an isomorphism of function fields. Hence, by Lemma 0.24, there exists $c \in \mathbb{C}[X]$ and a birational morphism $\phi : X_c \to Y$ such that χ is the localization of ϕ^*.

Q.E.D.

For Step (4), let $\phi : X \to Y$ be a morphism of affine algebraic sets with corresponding algebra homomorphism $\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$. Then if $x \in X$, $\phi^*(\mathfrak{m}_{\phi(x)}) \subset \mathfrak{m}_x$. In particular, we obtain a linear map $d\phi_x^* : T^*_{\phi(x)}(Y) \to T^*_x(X)$ given by $d\phi_x^*(f + \mathfrak{m}^2_{\phi(x)}) = \phi^*(f) + \mathfrak{m}_x^2$, which is called the codifferential. Its transpose $d\phi_x : T_x(X) \to T_{\phi(x)}(X)$ is called the differential of ϕ at x.

Note that if in addition $\psi : Y \to Z$ is a morphism, then $d(\psi\circ\phi)^*_x = d\phi^*_x \circ d\psi^*_{\phi(x)}$, since $(\psi\circ\phi)^* = \phi^*\circ\psi^*$. It follows that $d(\psi\circ\phi)_x = d\psi_{\phi(x)} \circ d\phi_x$.

Remark 0.27. *EXERCISE 3 If $\phi : X \to Y$ is an isomorphism of affine varieties, then for all $x \in X$, $d\phi_x : T_x(X) \to T_{\phi(x)}(Y)$ is an isomorphism of vector spaces. In particular, if X is smooth and Y is singular, there is no isomorphism between X and Y.*

Remark 0.28. *EXERCISE 4 Let $X = \mathbb{C}$ and let $Y = V(y^2 - x^3) \in \mathbb{C}^2$. Define $\phi : X \to Y$ by $\phi(b) = (b^2, b^3)$.*

(i) Let $\beta = (0,0) \in Y$. Compute $T_\beta(Y)$ and $T^*_\beta(Y)$.

(ii) Let $a = 0 \in X$ and compute $d\phi_a : T_a(X) \to T_\beta(Y)$ and ${d\phi_a}^* : T^*_\beta(Y) \to T^*_a(X)$.

(iii) Prove that X and Y are not isomorphic as affine varieties.

Remark 0.29. *EXERCISE 5; these are all good exercises.*

(i) Define $\phi : GL(n) \to \mathbb{C}$ by $\phi(g) = \det(g)$. For $g \in GL(n)$, use the fact that $GL(n)$ is open in $M(n)$ to identify $T_g(GL(n)) = T_g(M(n)) = M(n)$ and for $z \in \mathbb{C}$, identify $T_z(\mathbb{C}) = \mathbb{C}$. Prove that $d\phi_g(A) = Tr(A)$, the trace of A.

(ii) Let $i : Y \to X$ be the inclusion of a closed subset Y in an affine set X. Prove that if $y \in Y$, then $di_y : T_y(Y) \to T_x(X)$ is injective.

(iii) If $\phi : \mathbb{C}^n \to \mathbb{C}^m$ is a linear map, show that if $a \in \mathbb{C}^n$, we can identify $d\phi_a : T_a(\mathbb{C}^n) \to T_{\phi(a)}(\mathbb{C}^n)$ with ϕ.

We combine the preceding results to prove the following theorem, which is the main result of these notes.

Theorem 0.30. *Let X be an affine variety and let $\alpha \in X$. Then*

(1) $\dim(T_\alpha(X)) \geq \dim(X)$*;*

(2) The smooth locus of X is open and nonempty.

Proof : By Proposition 0.26, there is $c \in \mathbb{C}[X]$ and a birational morphism $\phi : X_c \to Y := V(f)$ to an irreducible hypersurface. By Proposition 0.17 of the notes on fiber dimension, there is a nonempty affine open set V of Y such that $U_1 := \phi^{-1}(V)$ is affine open in X_c and $\phi : U_1 \to V$ is an isomorphism of affine varieties. By Proposition 0.18, the set Y_r of smooth points of Y is open and dense. Since Y is irreducible, $V_r := Y_r \cap V$ is nonempty and V_r is smooth by Remark 0.16. Let $U = \phi^{-1}(V_r)$ and note that $\phi : U \to V_r$ is an isomorphism of affine varieties. By Exercise 3 above, it follows that U is smooth. Hence, $U \subset X_r$, where X_r is the set of smooth points of X_r, so in particular, X_r is nonempty. Since $Sing(X)$ is closed, (2) follows.

To establish (1), note that $X_r \subset S_d(X)$, and $S_d(X)$ is closed in X by Proposition 0.19. Since X is irreducible and X_r is open and nonempty, $\overline{X_r} = X$, so $X = S_d(X)$. This gives (1).

Q.E.D.

The assertion that $\dim(T_\alpha(X)) \geq \dim(X)$ means in some rough sense that around α, $\mathbb{C}[X]$ cannot be generated by fewer than $\dim(X)$ functions. We make this more precise below.

Lemma 0.31. *(COROLLARY TO NAKAYAMA'S LEMMA) Let R be a local ring with maximal ideal $\mathfrak{n}$ and let M be a finitely generated R-module and let $N \subset M$ be a R-submodule. If $M = N + \mathfrak{m} \cdot M$, then $M = N$.*

Proof : M/N is a finitely generated R-module and the hypothesis implies that $M/N = \mathfrak{m} \cdot M/N$. Hence, by Nakayama's Lemma, $M/N = 0$, so $M = N$.

Q.E.D.

Lemma 0.32. *Let R be an integral domain with maximal ideal $\mathfrak{m}$ and suppose R is $\mathbb{C}$-algebra and $R/\mathfrak{m} = \mathbb{C}$. Let $S = R - \mathfrak{m}$ and let $\mathfrak{n} = S^{-1}\mathfrak{m}$, the maximal ideal of the local ring $S^{-1}R$. Then the map $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ given by $y + \mathfrak{m}^2 \mapsto \frac{y}{1} + \mathfrak{n}^2$ is an isomorphism of R-modules.*

Proof : This is the same as the proof of Lemma 0.7.

Q.E.D.

Lemma 0.33. *For a point α in an affine variety X, let $R_\alpha = S^{-1}\mathbb{C}[X]$, where $S = R - \mathfrak{m}_\alpha$ and let $\mathfrak{m} = S^{-1}\mathfrak{m}_\alpha$. Let $f_1, \ldots, f_k \in R_\alpha$. Then $\mathfrak{m} = R_\alpha f_1 + \cdots + R_\alpha f_k$ if and only if $\mathfrak{m}/\mathfrak{m}^2$ is generated as a $R_\alpha/\mathfrak{m}$-vector space by the images of $f_1, \ldots, f_k$.*

One direction of this result is easy. The other direction is a consequence of Nakayama's Lemma. The vector space $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{m}_\alpha/{\mathfrak{m}_\alpha}^2$, so it follows that the dimension of $T_\alpha(X)$ is the same as the minimal number of generators for the R_α-module $\mathfrak{m}$.

By Lemma 0.32, $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}_\alpha/{\mathfrak{m}_\alpha}^2$, so that the Lemma implies that the dimension of $T_\alpha(X)$ is the minimal number of generators of the local ring R_α.

The local ring R_α with maximal ideal $\mathfrak{m}$ from Lemma 0.33 is called a regular local ring if the length of a maximal chain of prime ideals of R_α is $\dim(X)$. A regular local ring is an integral domain, and is integrally closed in its fraction field.

We state the following result from commutative algebra.

Proposition 0.34. *Let $\alpha \in X$ be a smooth point of an affine variety X. Then R_α is a regular local ring.*

A point $\alpha \in X$ is called normal if R_α is integrally closed. If $U = X_f \subset X$ is the affine open set defined by nonvanishing of f, then U is normal if and only if each point $\alpha \in U$ is normal in X. By Theorem 0.30 and Proposition 0.34, there is a nonempty open set $V \subset X$ such that if $\alpha \in V$, α is a normal point of X. Since every open set in X is a union of principal open sets, it follows that there is a principal open set of X that is normal. This gives an alternative proof of Proposition 0.19 from the notes on fiber dimension, but requires more commutative algebra