POD, LSE and Wavelet decomposition: Literature Review

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1 Introduction

The problem of turbulent fluid flow remains the outstanding unsolved problem of classical physics. Motivated by the fact that turbulence is the rule rather than exception in flows of technological importance, there has been extensive theoretical and experimental work to explain the underlying physical processes. Osborne Reynolds [1] was the first who in 1894 recognized the importance of studying turbulent flows and introduced the idea of decomposing the velocity into time mean and fluctuating components. The so-called Reynolds-averaged Navier-Stokes equation has traditionally been the starting point for many investigations.

Contrasting the idea that the turbulence is completely random, numerous experiments over the last decades have shown the existence of large-scale quasi-organized vortical structures in a variety of free and wall bounded turbulent flows. These so-called *coherent structures* (CS) appear to be dynamically important and play a key role in determining the macrocharacteristics of the flow such as mass, momentum energy and heat transports, entraining combustion, chemical reactions, drag and aerodynamic noise generation. Theodorsen [2] and Townsend [3] discussed an existence of organized motion (eddies) in turbulent shear layers. A variety of visualization techniques were applied in order to find and investigate a large-scale structures in flows. One of the first visualizations were done by Brown & Roshko [4]. They investigated a plane turbulent mixing layer using two gases with different refraction properties. Shadowgraphs have reveled a well-'organized' almost two-dimensional vortex-like coherent structures, convecting at nearly constant speed. Those structures were similar to the instability structures in a laminar shear layer, but have attributes typically associated with turbulent flow: 'randomness' and broad energy spectrum. Large-scale structures were found not changed by changes in Re number, it affects only in producing fine-scale turbulence, superimposed on the structures. A pairing process was suggested as a mechanism of creating the CS. Similar observations were made by Winant & Drowand [5] using dyes in water tunnel in the initial vortex layer at moderate Re numbers.

The importance of coherent structures in terms of their characteristics and dynamical roles was and still is under a serious discussion. Cantwell [18] in 1981 reviewed a research of organized motion in different types of turbulent flows, discussed possible ways like discrete vortex dynamics and large eddy simulation to investigate dynamics of coherent structures, their applications in transition and control of mixing. Hussain [35], [38], [34] have introduced a definition of coherent structure as a connected turbulent fluid mass with instantaneously phase-correlated vorticity over its spatial extent and emphasized vorticity as a characteristic measure of CS. He proposed a triple-decomposition of the flow into a mean flow, CS and incoherent turbulence, discussed a possible origins of CS, their time evolution as a strange attractor in phase space, called for importance of dynamics, profound practical significance, understanding of turbulent measurements and control.

In order to qualitatively describe these structures, a several experimental techniques were developed to extract CS. Conditional sampling techniques (Kaplan [6], Antonia [10]) use a conditional criterion in order to investigate structures. *A linear-stochastic estimation (LSE)* (Adrian [11]) uses 2-point second-order correlation functions in terms of conditional flow patterns, other techniques like *VITA* technique [29] were used to investigate a spatial shape of CS.

The disadvantage of all conditional techniques is the dependence on a conditional criteria. The first rigorous theoretical approach to investigating coherent structures was developed by John Lumley [13],[14] and is termed the *Proper Orthogonal Decomposition (POD).* This unconditional technique uses a second-order statistics in order to extract organized large-scale structures from turbulent flows. The mathematical basic of POD is a *Karhunen-Loève expansion* (Karhunen [16], Loève [17])

In 80's a new technique of extracting CS - *wavelet transformation* ([4]) was developed. This technique utilizes so-called *localized waveforms,* which possess local properties in both Fourier and physical spaces and appears as a generalization of Fourier transformation. It provides a local information about spectrum, scales and positions of events in flows. Because this technique does not require a priori any information about structures, now it is widely used in analysis of intermittent events and structures in turbulent flows.

Below three major techniques, POD, LSE and Wavelet Transformation will be discussed. Brief theories, as well as application to aerodynamical problems are provided. Also VITA, WAG and Windowed Fourier transformation techniques are briefly mentioned.

2 Proper Orthogonal Decomposition (POD)

2.1 Theory and Properties

The body of POD technique is Karhunen-Loève (KL) decomposition. The idea is to describe a given statistical ensemble with the minimum number of modes. Let $u(t)$ be a random generalized process with t as a parameter (spatial or temporal). We would like to find a deterministic function $\phi(t)$ with a structure typical of the members of the ensemble in some sense. In other words, a functional in the following form needs to be maximized :

$$(R(t,t'), \phi(t)\phi^*(t'))/(\phi(t), \phi(t)) = \lambda \geq 0 \quad (1)$$

where $R(t,t') = E\{u(t) \cdot u(t')\}$ is the autocorrelation function of $u(t)$ and $u(t')$, $E\{f(u)\}$ is the average or expected value of $f(u)$, $(\cdot, \cdot)$ is a scalar product and the asterisk denotes a complex conjugate. The classical methods of the calculus of variations gives the final result for ϕ, if $R(t,t')$ is an integrable function

$$\int R(t,t')\phi^*(t')dt' = \lambda\phi(t) \quad (2)$$

The solution of (2) forms a complete set of a square-integrable orthogonal functions $\phi_n(t)$ with associated eigenvalues λ_n. It was shown that any ensemble of random generalized functions can be represented by a series of orthonormal functions with random coefficients, the coefficients being uncorrelated with one another:

$$u = \sum_{n=1}^{\infty} a_n\phi_n \ , \ \ E\{a_n a_m\} = \delta_{nm}\lambda_m \quad (3)$$

These functions are the eigenfunctions of the autocorrelation with positive eigenvalues. The eigenvalues are the energy of the various eigenfunctions (modes). Moreover, since the modes were determined by maximizing λ (the energy of a mode), the series (3) converges as rapidly as possible. This means that it gives rise to an optimal set of basis functions from all possible sets.

If the averaging is performed in time domain, then $u(t, \mathbf{x})$ (here t is a time parameter) can be represented as follows:

$$u(t, \mathbf{x}) = \sum_{n=1}^{\infty} a_n(t)\phi_n(\mathbf{x}), \quad (4)$$

where a's are temporal coefficients and ϕ's are the spatial eigenfunctions or modes.

The transformation (2)-(3) is *POD transformation*.

Because of discretization of experimental data, a vector form of POD is widely used. In this case u becomes an ensemble of finite-dimensional vectors, the correlation function R is a correlation matrix and the eigenfunctions are called eigenvectors.

On practical grounds, (3) (or(4)) usually is represented only in terms of a *finite* set of functions,

$$u \approx u_L = \sum_{l=1}^{L} a_l\phi_l \quad (5)$$

Briefly, some important properties of POD:

1. The generalized coordinate system defined by the eigenfunctions of the correlation matrix is optimal in the sense that the mean-square error resulting from a finite representation of the process is minimized. That is for any fixed L:

$$R_L = \int_0^T [u(t) - \sum_{n=1}^{L} a_n\phi_n(t)]^2 dt \rightarrow min \quad (6)$$

   iff $\phi_n(t)$ are KL eigenfunctions of (2).

2. The random variables appearing in an expansion of the kind given by the equation (3) are orthonormal if and only if the orthonormal functions and the constants are respectively the eigenfunctions and the eigenvalues of the correlation matrix.

3. In addition to the mean-square error minimizing property, the POD has some additional desirable properties. Of these, the minimum representation entropy property and satisfaction of the continuity equations are worth mentioning.

4. Algazi and Sakrison [43] showed that Karhunen-Loéve expansion is optimal not only in terms of minimizing mean-square error between the signal and it's truncated representation (Property 1), but also minimizes a number of modes to describe the signal for a given error.

5. For homogeneous directions POD modes are Fourier modes [14].

In the case of large number of elements in second-order correlation tensor the direct method of finding the eigenfunctions becomes practically impossible. Sirovich [36] pointed out that the temporal correlation matrix will yield the same dominant spatial modes, while often giving rise to a much smaller and computationally more tractable eigenproblem - *the method of snapshots*. Mathematically, for a process $u(t,x)$ instead of finding a spatial two-point correlation matrix $R_{ij} = 1/M\sum_{m=1}^{M}[u(x_i,t_m)u(x_j,t_m)]$, where N is a number of spatial points and solving (2) ($N \times N$-matrix), one can compute a temporal correlation $M \times M$-matrix A_{mn} over spatial averaging,

$$A_{mn} = \frac{1}{M} \int_V u(x, t_m) u(x, t_n) dx, \qquad (7)$$

where M is number of temporal snapshots and calculate $\phi_i(x)$ from $u_m(x) = u(t_m, x)$ series as

$$\phi_i(x) = \sum_{n=1}^{M} b_{m,i} u_m(x) \qquad (8)$$

where $b_{m,i}$'s are the solutions of the equation $\mathbf{Ab} = \lambda \mathbf{b}$. Usually $M \ll N$ and the computational cost of finding ϕ's can be reduced dramatically. The method of snapshots also overcomes the difficulties associated with the large data sets that accompany more than one dimension.

The optimality of POD reduces the amount of information required to represent statistically dependent data to a minimum. This crucial feature explains the wide usage of POD, also known as Karhunen-Loéve expansion in a process of analyzing data.

In [50] the limitations of POD with temporal averaging were discussed. It was shown that in this case the analysis uses only information that is close to a particular final state of the system and thus cannot be used for the system which has a several final states. Also it was pointed out that the analysis de-emphasizes infrequent events, although they could be dynamically very important (burst-like events in a turbulent boundary layer). Alternative averaging techniques were proposed and shown to be more informative in terms of investigating the system dynamics. Delville [64] pointed out that POD technique can be treated as generalization of Fourier transform in inhomogeneous direction.

The Proper Orthogonal Decomposition was proposed by Lumley, 1967, [13]. The mathematical background behind POD is essentially Karhunen-Loève procedure Karhunen [16] (1946), Loève [17] (1955). But the method itself is known under a variety of names in different fields: Principal Component or Hotelling Analysis (Hotelling, 1953, [45]), Empirical Component Analysis (Lorenz, 1956, [44]), Quasiharmonic Modes (Brooks el at., 1988, [46]), Singular Value Decomposition (Golub and Van Loan, 1983, [47]), Empirical Eigenfunction Decomposition (Sirovich, 1987, [36]) and others. Closely related to this technique is *factor analysis*, which is used in psychology and economics (Harman, 1960, [48]).

From the mathematical point of view POD or KL expansion (2) is nothing else but a transformation which diagonalizes a given matrix $\mathbf{R}$ and brings it to a canonical form $\mathbf{R}$=$\mathbf{ULV}$, where $\mathbf{L}$ is a diagonal matrix. Therefore the roots of KL actually go into the middle of the last century. A review of the early history of KL expansion can be found in [51]. The mathematical content of KL procedure is therefore classical and is contained in paper by Schmidt [49] (1907).

In homogeneous directions POD reduces to Fourier-Stiltjes integral and gives Fourier modes in CS expansion. These function are not well-suited to describe compact CS. Lumley [23] proposed a noise-shot method to reconstruct CS in these homogeneous direction. Recently Berkooz et al. [39] have proposed a usage of wavelets on homogeneous directions.

Because of the large amount of computations required to find the eigenvectors, POD technique was virtually unused until the middle of the century. Radical changes came with the appearance of powerful computers and development of efficient algorithms to compute the eigenfunctions (method of snapshots, [36]). Now POD (KL expansion) is used extensively in the fields of detection, estimation, pattern recognition, and image processing as an efficient tool to store random processes, in system controls. Proper Orthogonal Decomposition is used in connection with stochastic turbulence problems (Lumley, 1967, [13]). In that context, the associated eigenfunctions can be identified with the *characteristic eddies* of the turbulence field [12]. A really good review of POD application in turbulence can be found in [15]. Brief theory of POD, as well as other techniques of extracting CS are presented.

2.2 Application to turbulent flows

POD extracts a complete set of orthogonal modes by maximizing the energy of the modes. Any member of statistical ensemble can be expanded on this set, and this series converges as fast as possible. Also POD gives the eigenvalues, corresponding to the kinetic energy of each mode. But the serious requirements to experimental equipment like an availability of big data storage and automated computer-controlled experiments to obtain a detailed second-order statistics from the flow have delayed the wide application of POD basically until the last decade of the century.

Historically the first attempts were applied to *turbulent boundary layers*. One of the reasons to look at the boundary layer was an existence of a strong peak of a production of turbulent energy on the outer edge of the viscous sublayer. Kline et al. [37] were first to observe streamwise vortex pairs in the boundary layer near the wall and bursting events associated with them. Bakewell et al.[22] using experimentally obtained 2-point correlation matrix, have sketched dominant structures within the wall region consisting of randomly distributed counter-rotating eddy pairs elongated in streamwise direction. They showed that the sublayer and the adjacent wall region play an active role in the generation and preservation of a turbulent shear flow. Nonlinear mechanism of vortex stretching suggests that linearized theories cannot provide an adequate description of the viscous sublayer. Later these structures were identified with burst-like events in the boundary layer. Cantwell [18] provided a detailed discussion of the research of turbulent boundary layer. Aubry et al. [19] were able to model a turbulent boundary layer by expanding the instantaneous filed in experimentally founded as a streamwise rolls eigenfunctions. By truncation of the series a low-dimensional system was obtained from Navier-Stokes equation via Galerkin procedure. This model represented the dynamical behavior of the rolls and was analyzed by methods of dynamical systems theory. The model captured major aspects of the ejection and bursting events associated with streamwise vortex pairs. This paper appeared to be the first to provide a reasonably coherent link between low-dimensional chaotic dynamics and realistic turbulent open flow system. Spatio-temporal three-dimensional structures in a numerically simulated transitional boundary layer ($Re_{\delta_2} = 283$) was examined by Rempfer & Fasel [65] using POD. Λ-shaped vortices were found to be the most energetic modes in the flow. They correspond to physically observed structures and resemble bursting events in fully turbulent boundary layers. POD modes was argued to correspond to physically existing structures, when the most energy is collected in a first mode.

Bounded Flows

Because of the experimental difficulties in obtaining second-order statistics from the flow, a number of computational simulation results were used to apply POD technique to. Moin & Moser [21] made direct numerical simulations of N-S equations in turbulent channel for 128 x 129 x 128 grid points and Re=3200. The shot-noise expansion proposed by Lumley [23] was used to find the spatial shape of the CS. A dominant eddy was found to keep 76% of the turbulent kinetic energy. Similar computational research was done by Ball et al. [20] for 24 x 32 x 12 grid points, Re=1500. Snapshot technique (Sirovich [36]) was applied in order to extract the modes. The eigenfunctions were found as rolls (the most energetic mode) or shearing motions. First 10 modes captured 50% of the turbulent energy. Dynamical behavior of structures was discussed. Rolls provide the mechanism for the transport in channel flows like turbulent 'bursts', while shearing modes were suggested to relate to the instabilities in the flow. Gavridakis [61] used results of numerical simulation through a square duct with $Re = 4800$ to get the correlation matrix. Two different structures were found after applying POD.

Sirovich & Park [25] performed numerical simulations of *Rayleigh-Benard convection* in a finite box for 17 x 17 x 17 grid points. POD technique, snapshot method and group symmetry considerations were applied to investigate CS. 10 classes of eigenfunction were discovered, first eigenmode being captured 60% of the energy (Howes et al [26]). Numerically simulated flow in a square duct ($Re = 4800$, grid 767 x 127 x 127) was analyzed by POD in [61]. Two different structures were found near 'wall domain' and 'corner domain'. Near wall it was a pair of high and low-speed streaks with the first POD mode seizing 28% of energy. In the corner domain a vortex pair with common flow toward the corner with 43% of the total energy was discovered. It was found that the energy in this region goes from the turbulence back to the mean flow. The flow in duct was found to have a greater degree of organization.

T*urbulent jets and shear flows*

Payne & Lumley [12] used experimentally obtained 2-point diagonal correlation tensor to extract CS in the turbulent wake behind a circular cylinder. Off-diagonal elements were found from a mixing layer assumptions. Glauser et al. [31] investigated a large-scale vortex ring-like structure in the mixing layer in axisymmetric jet (Re $\approx 110,000$) at $x/D = 3$ by POD and shot-noise decomposition. First 3 mode were found to contain almost all energy of the flow. Kirby et al. [32] used a snapshot method for 2-dimensional large-eddy simulation of axisymmetric compressible jet flow on 240 x 80 point grid for $Re \sim O(10^4)$. They found first 10 modes, which hold 94% energy. A similar approach was used by Kirby et al. [33] to investigate a simulated supersonic shear flow. Sirovich et al. [24] applied POD to the analysis of digitally imaged 2-dimensional gas concentration fields from the transitional region of axisymmetric jet. Here POD was proposed basically as a methodology for analyzing and treating large experimental and numerical data. Ishikawa et al [59] applied POD and wavelet technique to investigate coherent structures in a turbulent mixing layer. POD procedure revealed the first mode to be the most energetic one, while wavelet transform found low-frequency modulation of vorticity in this mode. Hilberg et al [63] investigated large-scale structures existed in a mixing layer in a narrow channel using a rake of probes. They applied both classical and snapshot POD techniques and were able to identify the largest 2-dimensional mode. Snapshot POD for a conditionally averaged velocity field showed that first two modes contain the largest portion of coherent vorticity. Delville [64] also considered plane shear layer to apply POD. Two rakes of hot-wire probes were used to extract correlation matrix from the flow. Vectorial version of POD was found to give a better presentation of structures in the flow. Different aspects and problems applying POD procedure were also discussed. An axisymmetric jet was investigated in [53]. POD modes for near-field pressure in inhomogeneous streamwise direction were obtained for a first time and observed as growing, saturating and decaying waves. Phase velocity for a first few modes near saturation point were found to be the same $U_p = 0.58U_j$. Characteristic structure was reconstructed using the shot-effect decomposition. It was observed that vortex pairings do not happen periodically and vortex tripling occur quite often. Citriniti [66] applied POD to a mixing layer in fully turbulent ($Re = 80,000$) axisymmetric jet. 138 hot wires were used to obtain the correlation data. POD modes were projected into instantaneous velocity field and a temporal dynamics of the temporal coefficients and their corresponding large-scale structures were investigated. Kopp et al [60] investigated large-scale structure in far field of a wake behind a cylinder ($x/d = 420$, $Re = 1200$). They used a rake of 8 x-wire probes to get instantaneous velocity field in cross-stream and spanwise directions. They applied POD technique to extract first modes and used first two of them as a template for Pattern Recognition (PR) technique. The structures found via PR and POD were similar and represented negative u-fluctuations with outward v-fluctuation in cross-stream direction. In spanwise direction a double roller structure was discovered. A flow field in lober mixer (enhancing mixing device) was considered by Ukeiley et al [52]. A rake of the probes in streamwise direction collected data on the correlation matrix. First POD modes were projected back to instantaneous flow filed to get information about multifractal nature of the flow.

POD also was in use in order to analyze turbulent-like mathematical equations. Chambers et al. [27] used the results of numerical Monte-Carlo simulation of a forced *Burger's equation* for different Re numbers. A number of modes were extracted from the numerical solution. The large-scale structures were found to be independent from Re number and exhibit viscous boundary layers near the walls. Sirovich & Rodriguez [28] examined *Ginzburg-Landau equation* in chaotic regime. Complete set of uncorrelated functions was extracted and used as a basis for the dynamical description of CS in the attractor set. First 3-mode representation was found to simulate the system's behavior quite good. (Rodriguez & Sirovich [29])

Taylor-Couette flow: Tangborn et al. [30]

POD-based modeling

Minimal number of POD modes make it useful to build low-order dynamical models for different applications. Using a truncated number of the POD modes as a basis and projecting them to Navier-Stokes equations allow someone to reduce PDE down to a low-order system of ODE's. The rest of modes are modelled by a dissipative term. Analysis of the obtained system can shed a light into a dynamics of the system. Aubry et al. [19] were able to model a turbulent boundary layer by expanding the instantaneous filed in experimentally founded as a streamwise rolls eigenfunctions. The model captured major aspects of the ejection and bursting events associated with streamwise vortex pairs. Aling et al [53] proposed POD-based technique to construct low-order dynamical models for rapid thermal systems. Gunes et al [57] applied POD technique to reconstruct a first six modes for numerical time-dependent transitional free convection in a vertical channel. A low-order system of ODE's based on the knowledge of the most energetic modes in the flow, was able to predict stable oscillations with correct amplitudes and frequencies. Same approach [58] was done to investigate a dynamics of transitional flow and heat transfer in periodically grooved channel. Ukeiley [67] applied POD technique in attempt to build a dynamical model of a turbulent mixing layer. This mode was based on experimental data from two 12-probe rakes placed in the flow. Two-dimensional spanwise structures along with streamwise vortical structures were extract form the flow. The model predicted correctly statistical distribution of energy in cross-stream direction.

It was pointed out by Lumley [23], that POD mode or 'characteristic eddy' represents coherent structure only if it contains a dominant percentage of energy. In other cases, POD modes do not actually extract shape of CS, but rather give an optimum basis to decompose the flow.

Poje & Lumley [62] proposed another technique to obtain information about structures in flow. They decomposed flow into mean, coherent modes and incoherent small-scale turbulence and applied energy method analysis [9] to find the most energy containing large-scale modes. The results compared well the with conventional POD technique.

Both POD and wavelets were considered as new tools in analyzing large-scale structures for wind engineering in [54]. Bienkiewicz et al [55] applied POD decomposition to analyze pressure forces acting on a roof. For this purpose 494 taps were used to obtain a spatial pressure correlation matrix.

Karhunen-Loève expansion is in use in pattern recognition (Ash & Gardner [40], Fukunaga [41]), stochastic processes (Ahmed & Goldstein [42]), meteorology; because of the optimality of the expansion K-L decomposition is used in data compaction and reduction.

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3 Linear Stochastic Estimation

3.1 Theory

Conditional sampling techniques are widely used to identify and describe coherent motions or structures in turbulent flows. They find the average value of some flow characteristics like velocity, pressure,temperature, when the prescribed event happens at one or several points. Below the flow characteristics will be noted as $\mathbf{u}(\mathbf{x},t)$, the event data vector (the given vector of variables with the associated event occurrence) - as $\mathbf{E}$, and the conditional average as $\langle\mathbf{u} \mid \mathbf{E}\rangle$. A simple example of the event is a realization in a point ($\mathbf{x}'$,t'), in which a flow characteristic lies close to a given vector $\mathbf{c}$ with a relatively small window $d\mathbf{c}$,

$$\mathbf{E} = \mathbf{u}(\mathbf{x}',t') \; when \; \mathbf{c} \leq \mathbf{u}(\mathbf{x}',t') < \mathbf{c} + d\mathbf{c} \tag{9}$$

For this case, the conditional average selects from the ensemble of all flow realizations the subset in which $\mathbf{u}(\mathbf{x}',t')$ is close to the prescribed vector $\mathbf{c}$, and extracts a typical structure of the flow field within this subset. Mathematically it is just a simple condition that assumes nothing about the underlying structure. During the experiment, the window should be big enough to produce samples with reasonable frequency and small enough to resolve properly the conditional average. In general, the event can include the conditions on $\mathbf{u}$ and any functionals of $\mathbf{u}$ (like the derivatives $\frac{\partial \mathbf{u}_i}{\partial \mathbf{x}_j}$) at one or more locations. Direct experimental conditional average measurements are limited to a rather simple one-dimensional flows and a small amount of 'conditions' and becomes extremely time-consuming and difficult for full three-dimensional flows and/or a large number of applied conditions. One of the common used technique to avoid this experimental problem and to find the conditional average $\langle\mathbf{u} \mid \mathbf{E}\rangle$ is the *Stochastic Estimation (SE)*. This method extracts structure by approximating an average field in terms of the event data at some given locations. In other words, SE reconstructs a flow field, corresponded to the structure by using a knowledge of the flow at some selected points in space and/or time. The approximation of $\mathbf{u}$ by stochastic estimation will be denoted by $\hat{\mathbf{u}}$.

When the elements of $\mathbf{u}$ and $\mathbf{E}$ are joint normally distributed, $\langle\mathbf{u} \mid \mathbf{E}\rangle$ is a linear function of $\mathbf{E}$ (Papoulis, [19]), but in general, $\langle\mathbf{u} \mid \mathbf{E}\rangle$ is a non-linear function of $\mathbf{E}$. Unfortunately, the turbulent events are strongly non-linear processes, which lead to a non-normal probability distributions. Therefore, $\langle\mathbf{u} \mid \mathbf{E}\rangle$ is a non-linear function of $\mathbf{E}$ for most flows. Adrian in 1977 [1] introduced a technique to estimate conditional averages for any arbitrary conditions. He proposed to expand $\langle\mathbf{u} \mid \mathbf{E}\rangle$ in a Taylor series about $\mathbf{E} = 0$ as

$$\hat{u}_i {=} \langle\mathbf{u}_i \mid \mathbf{E}\rangle {=} L_{ij}E_j + N_{ijk}E_jE_k + ... \tag{10}$$

(repeated subscripts are summed) and truncate this series at some degree (Adrian, [1], [2]). The unknown coefficient tensors $\mathbf{L}, \mathbf{N}$ ... can be determined if someone requires the mean-square error between the approximation and the conditional average to be minimal,

$$\langle[\langle u_i \mid \mathbf{E}\rangle - L_{ij}E_j - N_{ijk}E_jE_k - ...]^2\rangle \to \min$$

The minimization leads to the orthogonality principle, which states that the error must be statistically uncorrelated with each of the data (details)

$$\langle[\langle u_i \mid \mathbf{E}\rangle - L_{ij}E_j - N_{ijk}E_jE_k - ...]E_k\rangle = 0$$

The case where the series contains only the first order term, simple algebra leads to a set of linear algebraic equations for the estimation coefficients L_{ij}

$$\langle E_jE_k\rangle L_{ij} = \langle u_iE_k\rangle, \tag{11}$$

This Stochastic Estimation then is called *the Linear Stochastic Estimation (LSE),*

$$\hat{u}_i {=}, \; linear, \; estimate, \; of \langle u_i \mid \mathbf{E}\rangle = L_{ij}E_j, \tag{12}$$

where $L_{ij} = L_{ij}(\mathbf{x}, \mathbf{x}')$ and $\mathbf{x}'$ is the location of the event data. Cross correlation tensor $\langle E_jE_k\rangle$ between each of the event data and between the data and the quantity to be estimated $\langle u_iE_k\rangle$ must be obtained by independent means (Adrian, [1]). If $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ is the velocity field and the event data consists of velocity vectors $\mathbf{E} = \mathbf{u}' = \mathbf{u}(\mathbf{x}', t')$ at the location $\mathbf{x}'$, then $\langle E_jE_k\rangle = \langle u_ju_k\rangle$ is the Reynolds stress tensor and $\langle u_iE_k\rangle = \langle u_iu'_k\rangle = R_{ik}(\mathbf{x}, \mathbf{x}', t, t')$ is the two-point, second-order, space-time velocity correlation. Thus, knowledge of R_{ik} gives the ability to approximate any conditional averaged quantities. Note, that R_{ik} is *the unconditionally sampled tensor.* Thus, *LSE offers a powerful tool of reconstructing conditional averaged estimates based on the information from the unconditionally sampled two-point correlation tensor.*

In general, conditional sampling involves different types of events like single-point vectors, two-point vectors, local deformation tensors, multi-point vectors, space-time vectors, space-wavenumber events and so on. In this case the correlation tensor must include all possible cross-correlated values.

Several Advantages of the LSE technique are:

1. It requires only unconditional correlation functions, which is much easier to measure experimentally.

2. Once the estimation coefficients have been computed, they are independent of the conditional event data. Estimates for a number of different events could be easily evaluated.

3. The tensor information contained in the correlation functions is presented in the form of simple scalar or vector fields, which are more appropriate for an analysis.

4.It can be proved that the estimated fields satisfy the continuity equation, and they possess the correct length and/or time scales.

5. The procedure is applicable to all turbulent flows

Thus, the linear stochastic estimation exploits a second-order correlation functions between the given event data and the flow field at some selected points. This procedure establishes a simple link between conditional averages, the coherent structure that they may represent, and correlation functions.

One of the shortcomings of LSE is that the structure of the estimated flow pattern is independent of the magnitude of $\mathbf{u}(\mathbf{x}', t')$ (see (11)). It might be argued from the physical grounds that the flow structures obtained through LSE are associated with weak fluctuations only and for the case of strong fluctuations they may be different.

LSE, as well as POD, uses the cross-correlation tensor $\mathbf{R}$ to extract structures from the flow. The connection between LSE and POD is straightforward to find ([5]). Recall, that POD decomposes the flow into an infinite number of orthogonal modes $\phi_n(\mathbf{x}, t)$, for them to be found from the integral equation

$$\int\int \mathbf{R}(\mathbf{x}', \mathbf{x}; t', t)\phi_{\mathbf{n}}(\mathbf{x}', \mathbf{t}')\mathbf{dx}'dt' = \lambda_n\phi_n(\mathbf{x}, t)$$

The eigenmodes are used to decompose the flow field as

$$\mathbf{u}(\mathbf{x}, t) = \sum a_n\phi_n(\mathbf{x}, t), \tag{13}$$

The cross-correlation tensor itself can be presented in terms of the orthogonal functions as

$$\mathbf{R}(\mathbf{x}', \mathbf{x}; t', t) = \sum \lambda_n\phi_{\mathbf{n}}(\mathbf{x}', \mathbf{t}')\phi_{\mathbf{n}}(\mathbf{x}, \mathbf{t}), \tag{14}$$

Using(11), (13) and (14), the equation (12) can be rewritten as

$$\mathbf{u}(\mathbf{x}', t') = \mathbf{u}(\mathbf{x}, t)\sum \phi_{\mathbf{n}}(\mathbf{x}, \mathbf{t})\mathbf{f}(\mathbf{x}', t) \tag{15}$$

where $\mathbf{f}(\mathbf{x}', t) = \lambda_n\phi_n(\mathbf{x}', t)/\sum \lambda_n\phi_n^2(\mathbf{x}', \mathbf{t}')$ can be viewed as relative contribution or weight of each mode, $\phi_{\mathbf{n}}(\mathbf{x}, \mathbf{t})$, to the conditional average. Therefore, *LSE can be treated as a weighted sum of an infinite number of POD modes.* From here another potential weakness of LSE comes. LSE gives the representation of the coherent structures, associated with conditional estimates in terms of *one single flow pattern only.* It could lead to physically wrong conclusions when two or more distinctive structures exist in the flow. On the contrary, POD considered all the orthogonal modes.

The flow fields obtained from the type of conditional averages were referred by Adrian [2] to as 'conditional flow patterns', or, more briefly, '*conditional eddies*' in an effort to distinguish them clearly from physical coherent structures and characteristic modes $\phi_n(\mathbf{x}, t)$ obtained from POD technique.

3.2 Application of LSE and SE in turbulent flows

The validity and accuracy of the approximation of conditional averages of turbulent flows by LSE method have been investigated by numerous studies. First in 1979, Adrian [2] investigated the existence of structures in isotropic turbulence. The accuracy of LSE was evaluated by adding the quadratic term into (10). The results of LSE and second-order SE were found almost identical. The estimate predicted conditional eddies in the shape of large-scale vortex rings. Physical interpretation of these rings was discussed.

In [3] the validity and accuracy of LSE was investigated in details by comparing stochastic estimates to direct experimental conditional averages (9) measured in four different turbulent flows:

a) Axisymmetric shear layer with $Re_D = 390,000$. The conditional average $\langle u_1(\mathbf{x}, t+\tau) \mid u_1(\mathbf{x}, t)\rangle$ was measured on the centerline of the shear layer at $x/D = 2$ and 3. According to LSE, the coefficient $L_{11}(\tau) = R_{11}(\tau)/\sigma_1^2$ (σ_1^2 should be independent of the value of the conditional event. This LSE prediction was verified for several different values of $u_1(\mathbf{x}, t)$. Systematic differences was discovered for small values of τ, which questioned the accuracy of the linear estimates for small scales.

b) Plane shear layer ($Re_{\delta_\omega} = \Delta U_1\delta_\omega/\nu = 45,000$). The linear estimate of $\langle u_1(\mathbf{x}, t+\tau) \mid u_1(\mathbf{x}, t), u_2(\mathbf{x}, t)\rangle$ for the case of two conditional components of velocity at $\mathbf{x}$ became

$$\hat{u}(\mathbf{x}+\mathbf{r}, t) = L_{i1}u_1(\mathbf{x}, t) + L_{i2}(\mathbf{x}, t)u_2(\mathbf{x}, t), \quad i = 1, 2$$

Measurements of the conditional averages of u_1 and u_2 were compared to their linear estimates for the case $u_1 = 1.0\sigma_1$ and $u_2 = -1.0\sigma_2$ ($\sigma_{1,2}$ are RMS's of $u_{1,2}$ respectively). The linear estimate represented the large scale structure of the conditional averages with accuracy within the limits of measurement uncertainty.

c) Turbulent pipe flow ($Re = 50,000$). The quantity $\langle u_i(\mathbf{x}, t+\tau) \mid \mathbf{u}(\mathbf{x}, t)\rangle$ was measured experimentally and calculated by SE. Two velocity components were measured by X-film probes. The accuracy of LSE was evaluated by including the nonlinear terms in the stochastic estimation

$$\hat{u}_i(\mathbf{x}, t+\tau) = L_{ij}(\mathbf{x}, \tau)u_j(\mathbf{x}, t) + N_{ijk}(\mathbf{x}, \tau)u_j(\mathbf{x}, t)u_k(\mathbf{x}, t)$$

Also the conditional average $\langle u_i(\mathbf{x}+\mathbf{r},t) \mid \mathbf{u}(\mathbf{x},t)\rangle$ was measured using the two X probes. Again, the nonlinear estimate including a second-order term was found. In both cases, LSE estimates matched the experimental conditional averages really well. Second-order SE indicated relatively little improvement in accuracy versus the LSE,.

d) Grid turbulence with Reynolds number of 57,000 based on the free-stream velocity and the mesh spacing. The conditional average with time delay $\langle u_1(\mathbf{x},t+\tau) \mid u_1(\mathbf{x},t)\rangle$ and the autocorrelation function $R_{11}(\tau)$ were measured using a single hot-wire probe. The linear estimate and the experimental conditional average compared well for values of the event velocity $-1.5\sigma_1$, $-1.0\sigma_1$, $1.0\sigma_1$and $1.5\sigma_1$, where σ_1 is RMS of u_1.

The authors concluded that, although all considered turbulent flows were non-normal joint random processes, LSE worked well in these flows.

In [5] the optimal choice of a model of stochastic estimation was discussed. It was shown that the choice is strongly dependent on the event upon which the average is conditioned. The series of tests based on one-point velocity measurements in a turbulent boundary layer ($Re_\theta = 3,100$) demonstrated how the stochastic estimation may be refined to give more accurate descriptions of particular coherent motions. Selection of the order of the stochastic estimate was also discussed and it was shown that merely increasing the order of a polynomial model of the event vector would not necessarily increase the accuracy of the stochastic estimation.

Conditional eddies in isotropic turbulence were the focus of [9]. The velocity measurements for isotropic grid turbulence were used to approximate the stochastic estimate in a higher-order (up to a fourth-order) expansion. It was concluded the LSE is a acceptable method for investigating the qualitative large scale structures.

Fully turbulent boundary layer ($Re_\theta = 4900$) was under investigation in [14]. Measurements consisted of four simultaneously sampled hot-wire signals on a 13 x 11 x 8 three-dimensional grid. The results of linear and nonlinear estimates for two different events (fourth ($u>0, v<0$) and second ($u<0, v>0$) quadrant events) were compared to the corresponding experimental conditional averages and the agreement was found to be excellent. Extensions of the technique to space-time estimates of one- and two-point conditional averages were presented. It was pointed out that multipoint conditional averages focus on a particular scale based on the separation between the points at which conditions were imposed. Again, the stochastic estimation was found to allow one to implement averaging technique easily for a variety of conditions.

Multipoint conditional averages and spatio-temporal evolution of the three-dimensional structures of the turbulent wake of a cylinder was investigated in [17] (at $x/D = 100$ for $Re = 5000$). The measurements of the full 3-D correlation tensor for all components across the width of the wake were used to identify events contributing most to the Reynolds stress. The most likely ensemble averaged structure corresponding to these events were reconstructed using the stochastic estimation procedure. A new technique, the pseudo-dynamic reconstruction was developed to estimate the evolution of 3-D velocity fields from the experimental data.

In [6] multi-point conditional analysis obtained from LSE was applied to investigate dominant structures in a jet mixing layer. Since a single-point estimates did not give an adequate presentation of the dynamics of underlying structures, pseudo-dynamical evolution of the structures based on two-point velocity measurements was compared with the instantaneous velocity field. The agreement was found qualitatively good. It was noticed that in inhomogeneous flows a conditional sampling is pretty much a function of the location of the probes.

In [13] the results of various investigations based on stochastic estimation of the structures in turbulent shear flow were discussed. They supported a picture in which vortex rings and/or hairpin vortices dominate in the outer layer of the shear flow, whereas structures elongated in the streamwise direction with vorticity in the streamwise direction dominate inside the wall layer.

In [20] two-point space-time pressure-velocity correlations have been measured in the near field of a round jet flow with a stationary velocity probe and a moving pressure transducer. The correlations revealed the downstream translation of the vortices, and the physical mechanisms responsible for this behavior of the correlations were identified. Conditional averages of pressure fluctuations given the state of the velocity fluctuation was also obtained. A linear estimation mode agreed very well with the measured data.

[21] The properties of conditional averages have been studied for isotropic turbulence and for anisotropic turbulence in a plane high Reynolds number shear layer with 2:1 velocity ratio. It was shown that in isotropic turbulence a linear estimate was the dominant term. It gave a good approximation for estimated values of the velocity fluctuations, and it predicted a vortex ring structure in the flow. In the shear layer, conditional averages of the velocity component $u(x+r,t)$ and $v(x+r,t)$ and the Reynolds stresses $uv(x+r,t)$, $u^2(x+r,t)$, $v^2(x+r,t)$ have been calculated by measuring the values of $u(x,t)$ and $v(x,t)$ using two X-wire probes. Comparison of the conditional velocity field with its linear estimate showed a good agreement.

Adrian and Moin in [8] applied LSE to homoheneous turbulent shear flow data generated by a direct numerical simulation. Events contributing to the most of the Reynolds shear stress were thoroughly investigated. The conditional event under consideration was the value of the velocity $\mathbf{u}$ and the deformation tensor $d_{ij}(\mathbf{x}, t) = \frac{\partial u_i}{\partial x_j}$. The including of $\mathbf{d}$ was shown to have a significant influence on the conditional eddy field. A rational method of selecting events was proposed. It was based on determining the values of $(\mathbf{u}, \mathbf{d})$ at which the greatest contribution to some mean quantities occurred. It was found that one conditional-eddy structure was a hairpin vortex. Overall, LSE gave a good approximation of the conditional average on large scale.

In [4] LSE technique was applied to turbulent plasma to investigate the formation, propagation and decay of negative potential wells, which corresponded to ion phase-space vortices. Additional conditions as a sign of the derivatives of the velocity were imposed on the signal to improve the representation of extracted coherent structures in the turbulence. The results of LSE were found to be useful as a guideline to analyze the physics of turbulence.

Adrian in 1994 [24] made a detailed review of implementation of the stochastic estimation in a search of conditional structures. The mathematical method of SE was presented and the selected results obtained from different experiments in isotropic turbulence, turbulent boundary layer and channel flow were discussed and summarized.

3.3 Comparison of LSE (SE) with other techniques

Number of papers compared LSE (and/or SE) with the other existing techniques, as Proper Orthogonal Decomposition (POD), which as also based on the analysis of unconditional correlation tensor.

Sullivan and Pollard in [25] discussed methods for the eduction of coherent structures using multipoint measurements. They considered a three-dimensional wall jet using four different techniques: POD, LSE, Gram-Charlier Estimation (GCE) and Wavelet Decomposition (WD). The analysis of the obtained data revealed that the first POD mode clearly captured the local peaks in the velocity field. The comparison of both POD and LSE results revealed very good agreement, because LSE is actually a weighted sum of a infinite number of POD eigenmodes [5]. It was noticed that, *unless the first mode contains a significant amount of the total energy of the data, a flow structure may not be identified.* The GCE was used to develop a spatially dense mapping of the instantaneous velocity field. Combination of GCE and LSE helped to reconstruct evolution and interaction between structures downstream in the flow. These structures were shown to be associated with the exiting structure from the nozzle. Finally, wavelet decomposition was used to identify scales important within the flow. WD was found to provide a more general method for investigating a flow field.

In [16] both POD and LSE were used to identify structure in the axisymmetric jet shear layer ($Re = 110,000$) and in the 2-D mixing layer. The new complementary technique was introduced. This technique took projections of the estimated velocity field obtained from LSE onto the POD eigenmodes to obtain random coefficients a_n in (13). These estimated random coefficients were then used together with the POD eigenmodes to reconstruct the estimated velocity field. It was shown this complementary technique allows one to obtain time dependent information about the velocity field while greatly reducing the amount of instantaneous data. This approach can be useful to build and verify low dimensional dynamical systems models based on POD coefficients.

In [26] POD, and conditional sampling techniques were applied to a simple test functions and the results were compared to determine the strength and weaknesses of each approach. POD was found to be useful only for the signals where the energy consisted in a few modes. The physical interpretation of POD results was also discussed. Conditional sampling was found to be sensitive to the conditions chosen. None of the above methods was found of capable of analyzing or identifying the local small-scale structure of the flow. New methods for investigating complex structures based on fractural and wavelet analysis were presented. The application of the wavelet transform was shown to provide a simple functional description in terms of local length scales and the distribution of velocity and vorticity within turbulence structures.

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4 Wavelet Analysis

4.1 Theory

The wavelet transformation of the continuous signal $f(t) \in L^2(\mathbf{R})$ is defined the following way:

$$F_g(\tau, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(t) g^* \left(\frac{t-\tau}{a} \right) dt \qquad (16)$$

where parameter a is called *dilatation parameter,* τ is called *shift parameter,* asteric denotes a complex conjugate and the complex valued function $g(x)$ is called a *wavelet mother function* and satisfies the following conditions:

$$\int g(x) g^*(x) dx < \infty \qquad (17)$$

$$C(g) = 2\pi \int_{-\infty}^{+\infty} \frac{|\widehat{g}(\omega)|^2}{\omega} d\omega < \infty \qquad (18)$$

Here and below the hat sign over a function denotes a Fourier transformation of the function,

$$\widehat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-i\omega x} dx$$

The condition (18) is called *the admissibility condition* and in case of integrable function $g(x)$ implies that $\int_{-\infty}^{+\infty} g(x) dx = 0$. The admissibility condition guarantees the existence of *the inverse wavelet transformation*,

$$f(t) = \frac{1}{C(g)} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{F_g(\tau, a)}{\sqrt{a}} g \left(\frac{t-\tau}{a} \right) \frac{da\ d\tau}{a^2}$$

Because of the local support in physical domain (17), the integration in (16) is evaluated over a finite domain, proportional to a and centered near τ. This property of the wavelet transformation allows someone to analyze local characteristics of the signal at time τ and scale a.

The wavelet transformation (16) can be rewritten in the Fourier space as follows

$$F_g(\tau, a) = \sqrt{a} \int_{-\infty}^{+\infty} \widehat{f}(\omega) \widehat{g}^*(a\omega) e^{i\tau\omega} d\omega$$

and gives another highly useful interpretation of the wavelet transformation as a *multiple band-bass filtering* acting on the signal $f(t)$.

Parseval theorem - the energy of the signal can be decomposed in terms of the wavelet coefficients $F_g(\tau, a)$ as

$$\int_{-\infty}^{+\infty} f(t) f^*(t) dt = \frac{1}{C(g)} \int_0^{+\infty} \int_{-\infty}^{+\infty} F_g(\tau, a) F_g^*(\tau, a) \frac{da\ d\tau}{a^2}$$

Energy at the scale a is defined as

$$E(a) = \frac{1}{C(g)} \frac{1}{a^2} \int_{-\infty}^{+\infty} F_g(\tau, a) F_g^*(\tau, a) d\tau$$

4.2 Properties of wavelet transformation

1. Linear operator:

$$F_g(f_1 + f_2)(\tau, a) = F_g(f_1)(\tau, a) + F_g(f_2)(\tau, a)$$

2. Commutative with differentiation:

$$\frac{d^n}{dt^n} \{F_g(f)(\tau, a)\} = F_g \left(\frac{d^n f}{dt^n} \right) (\tau, a)$$

3. For any function $f(t)$ of homogeneous degree α at $t = t_0$, t.e. $f(\lambda t) = \lambda^\alpha f(t)$ near $t = t_0$:

$$F_g(f)(t_0, a) = a^{\alpha+1/2} F_g(f) \left(\frac{t_0}{a}, 1 \right)$$

$$or \quad F_g(f)(t_0, a) \sim a^{\alpha+1/2} \quad as \quad a \longrightarrow 0$$

This property could be really useful when analyzing a local regularity of the function $f(t)$. For instance, if the function is discontinuous at t_0, $\alpha = -1$.

4. The information provided by the complete set of the coefficients is *redundant*, t.e. there is a strong correlation between the wavelet coefficients,

$$F_g(\tau_0, a_0) = \int \int p \left(\frac{\tau_0 - \tau}{a}, \frac{a_0}{a} \right) F_g(\tau, a) \frac{da\ d\tau}{a^2},$$

$$where$$

$$p(\tau, a) = \frac{1}{C(g)} \frac{1}{\sqrt{a}} \int g^* \left(\frac{t-\tau}{a} \right) g(t)\ d\tau$$

Last property is not desirable when working with turbulence modelling, because the number of coefficients to be is very big. One can construct the orthonormal basis of the functions $\{\psi_{ij}(x)\}$, complete in $L^2(\mathbf{R})$ and orthogonal to themselves when translated by a discrete step and dilatated by a power of 2:

$$\psi_{ij}(x) = 2^{j/2} \psi(2^j x - i)$$

$$\int \psi_{ij}(x) \psi_{kl}^*(x) dx = \delta_{ik} \delta_{jl}$$

The *discrete version of the wavelet decomposition*

$$f(x) = \sum_i \sum_j F_{ij} \psi_{ij}(x) \ , \ where$$

$$F_{ij} = \int f(x) \psi(x - 2^{-j} i) dx$$

removes the redundancy and thus minimizes the number of the wavelet coefficients F_{ij} in L^2-norm to describe a given function $f(x)$. This transformation is attractive from computational point of view as an alternative candidate for the Fourier transformation of Navier-Stokes equations. Complete theory of discrete wavelet transform can be found in [5], [6]

From a variety of wavelet mother functions constructed by now, a few important wavelets are worth of mentioning:

Continuous wavelet transform

1. Morlet wavelet - good for an analysis of local periodicity of the signal:

$$g(x) = \exp(ibx)\exp(-x^2/2), \;\; b \geq 5$$

2. 'Mexican Hat' wavelet (Maar wavelet) - good for a search of localized structures:

$$g(x) = \frac{d^2}{dx^2}\exp(-x^2/2)$$

3. First derivative of a Gaussian: investigation of local gradients:

$$g(x) = \frac{d}{dx}\exp(-x^2/2)$$

Discrete wavelet transform

4. Lemarie-Meyer-Battle (LMB) wavelet - explicit definition [7]

5. Daubechies compactly supported wavelet - recurrent definition [5]

Generalization of the wavelet transformation to a multi-dimensional case is pretty straightforward ([30], for instance) and will be not discussed here.

Historically the idea of generating the basis which possesses a locality property in both physical and Fourier spaces goes to quantum mechanics [2] and signal processing [1]. In aerodynamics the similar constructions can be found in work by Siggia (1977) [32] and Zimin (1981) [18]. In attempt to build a model capable of predicting intermittent properties of small-scale turbulence, they proposed an algorithm of constructing a basis of *wave packets* which is local in Fourier and physical space. Independently Morlet applied a wavelet decomposition as a modification of Gabor elementary wavelets [1] in seismology [8], [37]. The first rigorous theory of wavelet decomposition was done by Morlet and Grossmann in 1984 [4]. Discrete version of the wavelet transform was developed by Daubechies [5], [6]. An excellent presentation of wavelet theory as well as it's application in turbulence research was done by Farge in [30]. Another good source is IEEE issue on wavelets [42].

4.3 Applications to turbulent flows

It is well-known that turbulence has both energy spectrum cascade, which is well-described in Fourier space and spacial intermittent events, localized in space and time ([38], e.g.). The wavelet property to be local in both physical and Fourier spaces looks really attractive to apply it in turbulence research. Original Kolmogorov's theory of turbulence [39] was modified to incorporate experimentally discovered properties of real turbulence as intermittency effects and energy backscatter (transfer of energy from small scales to large scales). A number of so called *hierarchical models* of 2-D and 3-D isotropic turbulence, based on the wavelet decomposition was built. One of he first attempt for 2-D turbulence was made in 1977 [32]. It utilized a *wave-packet* decomposition, which is essentially a wavelet transform. It was pointed out that the convection term in N-S equations is essentially local in physical space and the pressure term is local in wavenumber space and guarantees the incompressibility. Thus, the Fourier space was divided in number of *shells* $b^n < |\mathbf{k}| < b^{n+1}$, $n = 1..\ln_b \Lambda_K$. Here $b \sim 2$, $\mathbf{k}$ is a wavenumber in Fourier space and Λ_K is Kolmogorov dissipation scale. A wave-packets were introduced as a functions $\varphi_n(\mathbf{r})$ with their Fourier transform $\Phi_n(\mathbf{k})$ to have a non-zero value only within a corresponding shell. The functions φ_n are local in both Fourier and physical space. The Fourier decomposition in physical space within a shell was used to describe the functions. After projecting N-S equations into this basis the system of non-linear ODE's was obtained. After several simplifying assumptions, the interaction between adjacent shells was examined numerically for 4 shells x 26 Fourier mode decomposition within a shell. The energy cascade and temporal intermittency was investigated.

Independently in 1984 [18] essentially the same approach was taken to build a model of 3-D turbulence. A basis functions with a non-zero value within a shell were introduced. After the inverse transformation the physical representation was obtained. These *eddies* were distributed in physical space randomly and were allowed to move. After projecting N-S equations into this basis a system of non-linear ODE's was investigated. The model was found to have a qualitative agreement with experimental data. Later this model was modified using eddy representation in wavelet form in [19]. This model does not involve any empirical assumptions and possesses some important features of the turbulence, including $k^{-5/3}$ Kolmogorov spectrum. non-local interactions between the scales and back-scatter of energy from small scales to large scales. It can be used as a subgrid non-equilibrium turbulence modelling.

In [20] a *hierarchical-tree* model of 2-D turbulence was presented. The cascade of vortices was arranged in a tree-like structure, with smaller 'children' vortices connected to a bigger 'parent' vortex from the previous level. The main difference from shell-based models was that the distances between 'parent' and 'children' physical vortices were kept fixed. Thus a new variables for this Hamiltonian model were the amplitudes of vortices and the relative angles between adjacent vortices. Almost orthogonal basis of discrete wavelets was used. The model exhibits the spatial intermittency and fractal properties of it were obtained. The model showed good agreement with theoretically predicted energy cascade for 2-D turbulence.

One of the first application of the wavelets in turbulence was done by Farge et al [22], where an optimally constructed wavelet decomposition was proposed to reduce a number of basis functions (modes) to describe turbulence evolution. Investigating an evolution of a truncated subset of important modes, they have shown better performance of the wavelet decomposition against the standard Fourier decomposition. Another classical paper in this area is [30], where a brief theory and the basic properties of the wavelets and their application in turbulence is well-presented.

In [11] the wavelet transformation was applied for analysis of turbulent flows. It was pointed out that the wavelets closely resemble the 'eddy' structures introduced by Lumley [3] as a building blocks for turbulence. The brief theory and basic properties of the continuous wavelet transformation was presented. In an example of shock wave/free turbulence it has been clearly showed the advantage of wavelet transform to detect discontinuities of a signal (front of a shock-wave). Thus, intermittent events can be easily localized using 'Mexican hat' wavelets. Another example concerning the wall turbulence was considered. The wavelet technique was compared with VITA technique [29] for a detection of sweep and ejection events taking places in the boundary layer. It was found that VITA technique could give false or no detections of these burst-like events and requires some knowledge about the characteristic scales of the events. It comes from the fact that VITA technique is *a single scale filtered technique*. Because of variable scaled filtered property of the wavelet transformation it detects all the events well and does not require any *a priory information* of the events to be detected. Also a discretized version of the wavelet transformation was proposed as a competitive technique for CFD versus Fourier based technique for flows with large local gradients or discontinuities.

In [12] coherent structures (ejections and sweeps) in a heated turbulent boundary layer ($Re = 5124$ based on the momentum thickness $\delta = 6.1\ mm$) were investigated. Regions near the viscous sublayer $y^+ = 7$ and a fully turbulent central region $y^+ = 180$ were considered and both velocity and thermal measurements were taken in these regions using cold and hot wires. Several techniques like VITA, WAG (Window Averaged Gradients)[28] and wavelet analysis based on Morlet mother function were used to analyze the intermittent events. Conditional spectra for cooling (ejection) and heating (sweeps) events were obtained. The ejections were found to be relatively slow more localized events with comparison to more rapid sweep events.

In [15], [16] a discretized version of the three-dimensional wavelet transformation using LMB wavelet was applied for turbulent flows, both experimental (wake behind a cylinder) and numerically simulated ones. Spatially dependent quantities like total kinetic energy $E(k,x)$, net transfer to the wavenumber k, $T(k,x)$ and total flux through the wavenumber k to all smaller scales, $\pi(k,x)$ were derived using wavelet analysis. Analysis of these quantities from experimental data have shown a significant level of spatial intermittency with large variations from the mean values, essentially at a smaller scales. Spatial pdf's for energy distribution and dissipation were obtained and revealed an existence of long exponential tails. Several versions of the wavelet mother function were used to verify a robustness of the results. The wavelet results has revealed a multifractal nature of the turbulence at small scales and some fractal statistics (the generalized dimensions) of the multifractal turbulence were calculated. An additive mixed multifractal cascade model was built and was shown to duplicate all the essential results of the experiments. Also a multifractal behavior of turbulence was explored in [38].

Similar approach of analyzing generalized dimensions of turbulence has been taken in [33]. Again, turbulence was treated as a multifractal process and the continuous wavelet transformation was applied to the equations of motion. The author was able to come up with the system of dynamical equations for an evolution of the generalized dimensions based on the dynamics of Navier-Stokes equations. Thus, the model which possesses both multifractal thermodynamical properties and the dynamics inherent to N-S equations was proposed.

An application of the discrete wavelet transform to explore the intermittency in turbulent flows was a purpose of the work [34]. Authors have been established a firm connection between the intermittent events and underlying coherent structures. Using the experimental data for isotropic turbulence behind a grid, they found that the flow reveals a big degree of intermittency for $Re_\lambda \geq 10$. Also applying a velocity phase averaged technique based on the wavelet transform, they have been able to reconstruct the 'signature' of the coherent structures corresponding to the intermittency events. For an isotropic turbulence a typical size of the small-scale filamentary-like structures have been found as of $(4 \div 5)$ Kolmogorov scales , with these structures capturing about 1% of the flow energy. For a jet-generated turbulence ($Re_\lambda = 250 \div 800$) large-scale structures (vortex rings) from a longitudinal component of velocity as well as small-scale structures (filamentary structures) from a transverse component were found. A strong phase correlation between the large and small scale structures was observed. The universality of a scaling exponents $\xi(p)$ in Kolmogorov's scaling law $|U(x+r) - U(x)|^p \sim r^{\ \xi(p)}$ for $p = 2..6$ was verified for a grid and jet turbulent flows for moderate Re_λ.

A transition to turbulence in a shear layer was investigated using the continuous wavelet transform in [24]. A pairing process was found to be intermittent in the region where the subharmonic dominates over the fundamental mode, with the intermittency being stronger for large scales of motion.

In [17] flow behind a sphere in a stratified fluid was analyzed for a range of Re and Fr numbers. The DPIV technique was used to get the velocity field. The two-dimensional version of Morlet wavelet was applied to obtain spatially local length scales, as well as local Re and Fr numbers. Spatial distributions of Re and Fr were found to be similar. Vortex core centers were marked by very low Re and Fr numbers.

The wavelet decomposition turns out to be an optimum tool to analyze this object. In [14] 2-D turbulence from an axisymmetric jet with $Re_D = 4,000$ was investigated. A two-dimensional version of 'Mexican Hat' wavelet was used to decompose dye concentration pictures. Two different structures, large-scale *beads* and small-scale *strings* were observed. Strings were found to be a strongly anisotropic structures with essential lack of self-similarly across the scales.

In [24] a continuous wavelet analysis was applied to investigate transition in turbulent mixing layer. It revealed that the paring process is intermittent. In [26] intermittent events on different scales in atmospheric wind were reported by applying the wavelet technique. In [40], [41] several techniques like wavelet transform, VITA and WAG were applied to wind velocity data . Intermittent coherent large and small structures were discovered and investigated. Dynamics of sweep-ejection process was discussed and the obtained results were found in agreement with previous results. In [43] a shear layer in a region of strong interaction between a fundamental and subharmonic modes was investigated by a continuous Morlet wavelet transform. An intermittent π-shifts in subharmonic phase were discovered. Motivated by the experimental data, a dynamical Hamiltonian model based on structural interactions of vortices was proposed and it was shown that the model have a very good agreement with the experiment.

A multiple acoustic modes in an underexpended supersonic rectangular jet were investigated by the wavelet transform. in [44]. They found to exist simultaneously and don't exhibit a mode switching. Rather detailed wavelet-based analysis of acoustic mode behavior in supersonic jets can be found in [45]

In [27] new methods for investigating complex structures based on fractural and wavelet analysis were presented. The application of the wavelet transform was shown to provide a simple functional description in terms of local length scales and the distribution of velocity and vorticity within turbulence structures. In [25] several techniques for analysis of turbulent flows were discussed. The wavelet transform was found useful to detect intermittent events and turbulent structures in flows.

Wavelet transform was successfully used to diagnostics in seismology [37], cardiology [21], in diagnostics of engine cylinders [23]. In optics Optical version of Wavelet Transform (OWT) was introduced in [10]. Also the wavelet transform is actively used in multifractal analysis [35], [36] and information theory [7].

4.4 Comparison with other techniques

4.4.1 Windowed Fourier transformation

The main problem with Fourier transform is that the basis functions $\exp(i\omega t)$ do not belong to $L^2(\mathbf{R})$. In other words, they have an infinite span in physical space and provide no information about spacial localization of the signal. This problem could be overcome, if the signal is *windowed* or multiplied by a function with local physical support $w(t)$. This idea was introduced by Gabor in 1946 [1]. The windowed transformation is as follows

$$L(\tau,\omega) = \int f(t)\overline{w}(t-\tau)e^{-i\omega(t-\tau)}dt \qquad (19)$$

In practice, two kinds of $w(t)$ are widely used, cosine window $cos(\pi/2\ t)$, $t \in [-1,1]$ and Gabor packet $\exp(-t^2/2\sigma^2)$. Transform (19) resembles (16), but provides *a fixed time window* for any ω. This time window can be varied by an appropriate dilatation of $w(t)$, though.

4.4.2 VITA (Variable Integral Time Averaging)

VITA technique was introduced in 1976 [29]. The idea is to look for a short-time averaging version of RMS of the signal,

$$\begin{aligned} var(t,T) &= \frac{1}{T}\int_{t-T/2}^{t+T/2} f(t)^2 dt - \left(\frac{1}{T}\int_{t-T/2}^{t+T/2} f(t)dt\right)^2 \\ \sigma_f &= \lim_{T\to\infty} var(t,T)\ -\ RMS\ of\ the\ signal \end{aligned}$$

The event is said to be detected if $var(t,T) > K \cdot \sigma_f$

The technique introduce two constants to play with, T and K. This technique is valid only for *second-order* *stationary signals.* It uses a fixed time window, which means *low-pass single scale filtering.*

Problems with VITA

1. Choice of T and K is intricate
2. Does NOT detect events separated by time less than T
3. Possible false detections
4. Smoothing of some events

4.4.3 WAG (Window Averaged Gradients)

Another technique to detect sudden changes in velocity signal was proposed in [28] and it is based on the analysis of the average gradients.

$$WAG(t,T) = \frac{1}{\sigma_f}\frac{1}{T}\left[\int_{t-T/2}^{t} f(t)dt - \int_{t}^{t+T/2} f(t)dt\right]$$

The event is detected when $WAG(t,T) > \beta \cdot u'_{RMS}$. Similar problems as with VITA technique: single scale filtering, stationary signals only.

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