Lie groupoids and Lie algebroids

Songhao Li

Washington University in St. Louis
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- Lie algebroid
- Examples of Lie algebroids

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- Integrations of a Lie algebroid
- Examples of integrations
A Lie groupoid $\mathcal{G}$ over the base manifold $M$ is a category such that
- the set of objects is $M$, and the set of arrows $\mathcal{G}$ is a manifold;
- the arrows are invertible;
- the source $s$ and target $t$ are submersions;
- the multiplication $m$ and the identity id are smooth.
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- the arrows are invertible;
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- the multiplication $m$ and the identity $\text{id}$ are smooth.

The structure maps are summarized by the following commutative diagram:

$$
\mathcal{G}_s \times_t \mathcal{G} \xrightarrow{\text{id}} \mathcal{G} \xrightarrow{\text{id}} M
$$
Source-simply-connected Lie groupoid

**Notation**

For a Lie groupoid $\mathcal{G}$ over $M$, we denote it by $\mathcal{G} \rightrightarrows M$.

**Source-simply-connected Lie groupoid**

For a Lie groupoid $\mathcal{G} \rightrightarrows M$,

- for $x \in M$, the source fiber of $x$ is $s^{-1}(x)$, and the target fiber is $t^{-1}(x)$;

- $\mathcal{G} \rightrightarrows M$ is source-connected, if $s^{-1}(x)$ is connected for each $x \in M$;

- $\mathcal{G} \rightrightarrows M$ is source-simply-connected, if $s^{-1}(x)$ is connected and simply-connected for each $x \in M$. 
Examples: Lie groupoids

1. Lie group

If the base $M$ is a point, then $G$ is a Lie group.
Examples: Lie groupoids

1. Lie group
If the base $M$ is a point, then $\mathcal{G}$ is a Lie group.

2. Bundle of Lie groups
If the source $s : \mathcal{G} \to M$ and the target $t : \mathcal{G} \to M$ coincide, then $\mathcal{G} \rightrightarrows M$ is a bundle of Lie groups.
Examples: Lie groupoids

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3. Pair groupoid
   For a connected manifold $M$, we define the pair groupoid $\text{Pair}(M) \rightrightarrows M$:
   - $\text{Pair}(M) = M \times M$;
   - $s: \text{Pair}(M) \to M$, $(x, y) \mapsto x$;
   - $t: \text{Pair}(M) \to M$, $(x, y) \mapsto y$;
   - $\text{id}: M \to \text{Pair}(M)$, $x \mapsto (x, x)$;
   - $m: \text{Pair}(M)_s \times_t \text{Pair}(M) \to \text{Pair}(M)$, $((x, y), (y, z)) \mapsto (x, z)$. 
Examples: Lie groupoids

4. Fundamental groupoid

For a path-connected manifold $M$, we define the fundamental groupoid $\Pi_1(M) \rightrightarrows M$:

- $\Pi_1(M) = \{[\gamma] | \gamma : I \rightarrow M\}$ is the homotopy classes of paths;
- $s : \Pi_1(M) \rightarrow M$, $[\gamma] \mapsto \gamma(0)$;
- $t : \Pi_1(M) \rightarrow M$, $[\gamma] \mapsto \gamma(1)$;
- $id : M \rightarrow \Pi_1(M)$, $x \mapsto id(x)$, where $id(x)$ is the constant path at $x$;
- $m : \Pi_1(M)_s \times_t \Pi_1(M) \rightarrow \Pi_1(M)$ is the concatenation of paths.
Examples: Lie groupoids

4. Fundamental groupoid

For a path-connected manifold \( M \), we define the **fundamental groupoid** \( \Pi_1(M) \xrightarrow{} M \):

- \( \Pi_1(M) = \{[\gamma] \mid \gamma : I \to M\} \) is the homotopy classes of paths;
- \( s : \Pi_1(M) \to M, \quad [\gamma] \mapsto \gamma(0) \);
- \( t : \Pi_1(M) \to M, \quad [\gamma] \mapsto \gamma(1) \);
- \( id : M \to \Pi_1(M), \quad x \mapsto id(x) \)
  where \( id(x) \) is the constant path at \( x \);
- \( m : \Pi_1(M)_s \times_t \Pi_1(M) \to \Pi_1(M) \) is the concatenation of paths.

**Remark**

Note that \( (s, t) : \Pi_1(M) \to \text{Pair}(M) \) is a Lie groupoid morphism.
Examples: Lie groupoids

5. General linear groupoid

For a vector bundle $E \rightarrow M$, we define the \textit{general linear groupoid} $\mathcal{GL}(E) \rightrightarrows M$:

- $\mathcal{GL}(E) \simeq \{(x, y, \phi_{xy}) \mid x \in M, \ y \in M\}$
  where $\phi_{xy} : E_x \sim E_y$ is an isomorphism of vector spaces;
- $s : \mathcal{GL}(E) \rightarrow M$, $(x, y, \phi_{xy}) \mapsto x$;
- $t : \mathcal{GL}(E) \rightarrow M$, $(x, y, \phi_{xy}) \mapsto y$;
- $\text{id} : M \rightarrow \mathcal{GL}(E)$, $x \mapsto (x, x, \text{id}_{xx})$;
- $m : \mathcal{GL}(E)_s \times_t \mathcal{GL}(E) \rightarrow \mathcal{GL}(E)$,
  $((x, y, \phi_{xy}), (y, z, \phi_{yz})) \mapsto (x, z, \phi_{yz} \circ \phi_{xy})$. 
Examples: Lie groupoids

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- $t : \text{GL}(E) \to M$, $(x, y, \phi_{xy}) \mapsto y$;

- $\text{id} : M \to \text{GL}(E)$, $x \mapsto (x, x, \text{id}_{xx})$;

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  $((x, y, \phi_{xy}), (y, z, \phi_{yz})) \mapsto (x, z, \phi_{yz} \circ \phi_{xy})$.

Remark

A Lie groupoid action of $\mathcal{G} \rightrightarrows M$ on a vector bundle $E \to M$ is a Lie groupoid morphism $\rho : \mathcal{G} \to \text{GL}(E)$. 
Examples: Lie groupoids

6. Holonomy groupoid

For a vector bundle $E \to M$, we define the **holonomy groupoid** $\text{Hol}(E) \rightrightarrows M$:

- $\text{Hol}(E) = \{([\gamma], \phi_{\text{st}}) | [\gamma] \in \Pi_1 D.$ where $\phi_{\text{st}} : E_{\gamma(0)} \sim \rightarrow E_{\gamma(1)}\}$ is an isomorphism of vector spaces;

- $s : \text{Hol}(E) \to M, \; ([\gamma], \phi_{\text{st}}) \mapsto \gamma(0)$;

- $t : \text{Hol}(E) \to M, \; ([\gamma], \phi_{\text{st}}) \mapsto \gamma(1)$;

- $\text{id} : M \to \text{Hol}(E), \; x \mapsto (\text{id}(x), \text{id}_{xx})$;

- $m : \text{Hol}(E)_s \times_t \text{Hol}(E) \to \text{Hol}(E), \; ((\gamma, \phi_{\text{st}}), (\sigma, \psi_{\text{st}})) \mapsto (\gamma \circ \sigma, \phi_{\text{st}} \circ \psi_{\text{st}})$. 

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Examples: Lie groupoids

6. Holonomy groupoid

For a vector bundle $E \to M$, we define the **holonomy groupoid** $\text{Hol}(E) \rightrightarrows M$:

- $\text{Hol}(E) = \{(\gamma, \phi_{st}) | \gamma \in \Pi_1 D$.
  where $\phi_{st}: E_{\gamma(0)} \to E_{\gamma(1)}\}$ is an isomorphism of vector spaces;

- $s: \text{Hol}(E) \to M, \ (\gamma, \phi_{st}) \mapsto \gamma(0);$  

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  $((\gamma, \phi_{st}), (\sigma, \psi_{st})) \mapsto (\gamma \circ \sigma, \phi_{st} \circ \psi_{st}).$

**Remark**

There is an obvious surjective groupoid morphism $\text{Hol}(E) \to \text{GL}(E)$ which covers $\Pi_1(M) \to \text{Pair}(M)$. 
Examples: Lie groupoids

7. Gauge groupoid
For a principal $G$-bundle $\pi : P \to M$, we consider the diagonal action of $G$ on $P \times P$:

$$g(u, v) = (gu, gv).$$

We define the \textit{gauge groupoid} $Gg(P) \rightrightarrows M$:

- $Gg(P) = (P \times P)/G$.

For $u, v, v', w \in P$ such that $\pi(v) = \pi(v') = x$,

- $s([u, v]) = \pi(u)$;
- $t([u, v]) = \pi(v)$;
- $id(x) = ([v, v])$;
- $m([u, v], [v', w]) = ([u, w])$. 
Lie algebroid

Lie algebroid

A Lie algebroid $A$ over the base manifold $M$ is a vector bundle $A \to M$ with a Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$ and an anchor map $a : A \to TM$ that preserves the bracket and satisfies the Leibniz rule

$$[X, fY] = f[X, Y] + a(X)(f)Y.$$  (2.1)
Lie algebroid

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Lie functor

For a Lie groupoid $\mathcal{G} \rightrightarrows M$, the vector bundle

$$\text{Lie}(\mathcal{G}) \doteq \ker(Ts : T\mathcal{G} \to TM) |_{\text{id}(M)}$$  \hfill (2.2)

with the bracket on left invariant vector fields, and the anchor $Tt : \text{Lie}(\mathcal{G}) \to TM$, is Lie algebroid.
Examples: Lie algebroids

1. Lie algebra

If the base $M$ is a point, then $A$ is a Lie algebra.
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1. Lie algebra
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If the anchor $a : A \rightarrow TM$ is trivial, then $A$ is a bundle of Lie algebras.
Examples: Lie algebroids

1. Lie algebra
   If the base $M$ is a point, then $A$ is a Lie algebra.

2. Bundle of Lie algebras
   If the anchor $a : A \to TM$ is trivial, then $A$ is a bundle of Lie algebras.

3. Tangent algebroid
   - For a manifold $M$, the tangent bundle $TM$ itself is a Lie algebroid.
   - Both the pair groupoid $\text{Pair}(M)$ and the fundamental groupoid $\Pi_1(M)$ integrates the tangent algebroid $TM$. That is,
     \[ \text{Lie}(\text{Pair}(M)) = \text{Lie}(\Pi_1(M)) = TM. \]
Examples: Lie algebroids

4. General linear algebroid

For a vector bundle \( \pi : E \to M \) of rank \( r \), we define the \textit{general linear algebroid} \( \mathfrak{gl}(E) \):

- Let \( E \) be the Euler vector field, we have
  \[
  \Gamma(\mathfrak{gl}(E)) = \{ X \in \Gamma(TE) \mid \mathcal{L}_E X = 0 \}.
  \]

- The anchor \( a : \mathfrak{gl}(E) \to TM \) is defined by \( \pi_* : TE \to TM \). A section \( X \in \Gamma(\mathfrak{gl}(E)) \) is invariant under the fiber rescaling of \( E \to M \), so
  \[
  a(X) = \pi_*(X) \in \Gamma(TM)
  \]
  is well-defined.

- The sections \( \Gamma(\mathfrak{gl}(E)) \) are the derivations of \( E \).
Examples: Lie algebroids

Remark

- The general linear algebroid $\mathfrak{gl}(E)$ fits into the following exact sequence

$$0 \to V \to \mathfrak{gl}(E) \overset{a}{\to} TM \to 0 \quad (2.4)$$

where $V$ is the vector bundle whose sections are the vertical vector fields, i.e. vector fields on $E$ that are tangent to the fibered of $E \to M$.

- Both the holonomy groupoid $\mathcal{H}o\mathcal{l}(E)$ and the general linear groupoid $\mathcal{G}L(E)$ integrate the general linear algebroid $\mathfrak{gl}(E)$.

- A Lie algebroid action $A \to M$ on a vector bundle $E \to M$ is a Lie algebroid morphism from $A$ to $\mathfrak{gl}(E)$. 
Examples: Lie algebroids

5. Atiyah algebroid
For a principal $G$-bundle $P \to M$, the Atiyah algebroid $\mathbb{A}_t(P)$ of $P$ is the Lie algebroid of the gauge groupoid $\mathbb{G}_g(P) \rightrightarrows P$, which fits into the following short exact sequence:

$$0 \to P \times_{\mathbb{G}_g} \to \mathbb{A}_t(P) \to TM \to 0 \quad (2.5)$$

where $P \times_{\mathbb{G}_g} \mathbb{G}_g$ is the associated bundle.
Integrations of a Lie algebroid

- In general, it is not always true that a Lie algebroid integrates to a Lie groupoid. The integrability condition was given in [Crainic-Fernandes, Ann. Math. 2003].

- For an integrable Lie algebroid $A$, there is a unique, up to isomorphism, source-simply-connected groupoid $\mathcal{G}^{ssc}$ integrating $A$.

- For another Lie groupoid $\mathcal{G} \rightrightarrows M$ integrating $A$, there is a groupoid morphism $\mathcal{G}^{ssc} \rightarrow \mathcal{G}$.

- In some cases, there also exists an adjoint groupoid $\mathcal{G}^{adj} \rightrightarrows M$ that receives a map from other integration of $A$. 
Integrations of a Lie algebroid

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- For an integrable Lie algebroid $A$, there is a unique, up to isomorphism, source-simply-connected groupoid $G^{ssc}$ integrating $A$.
- For another Lie groupoid $G \Rightarrow M$ integrating $A$, there is a groupoid morphism $G^{ssc} \to G$.
- In some cases, there also exists an adjoint groupoid $G^{adj} \Rightarrow M$ that receives a map from other integration of $A$.

1. Lie groups

For a semi-simple Lie algebra $\mathfrak{g}$, we have the simply-connected Lie group $G^{sc}$, and the adjoint Lie group $G^{adj}$.
Examples of integrations

2. Tangent integrations

- For a path-connected manifold $M$, the integrations of the tangent algebroid $TM$ is given by normal subgroups of the fundamental group, $\Lambda(\pi_1(M, x))$.
- The equivalence is given by $G \mapsto M \mapsto t^*[\pi_1(s^{-1}(x), \text{id}(x))]$. (3.1)
- The fundamental groupoid $\Pi_1(M)$ is the source-simply-connected integration, and corresponds to the trivial group $\{1\} < \pi_1(M, x)$.
- The pair groupoid $\text{Pair}(M)$ is the adjoint integration, and corresponds to $\pi_1(M, x)$. 
Examples of integrations

3. Log tangent algebroid
Let $L$ be a closed hypersurface of $M$. The log tangent algebroid

$$T(M, \log L)$$

is a Lie algebroid whose sections are the vector fields tangent to $L$. 
Examples of integrations

3. Log tangent algebroid
Let $L$ be a closed hypersurface of $M$. The log tangent algebroid

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- In this case, there exists an adjoint groupoid integrating $T(M, \log L)$, which we call the log pair groupoid.
Examples of integrations

- Take the pair groupoid $\text{Pair}(M) = M \times M$, and the subgroupoid $\text{Pair}(L) = L \times L$. We blow up $\text{Pair}(M)$ along $\text{Pair}(L)$ and obtain the blow-down map

  $$\rho : \text{Bl}_{\text{Pair}(L)}(\text{Pair}(M)) \to \text{Pair}(M).$$

  Note: this is the real projective blow-up.

- From $\text{Bl}_{\text{Pair}(L)}(\text{Pair}(M))$, we remove the closures of $p^{-1}(L \times M)$ and $p^{-1}(M \times L)$, i.e. the proper transforms of $s^{-1}(L)$ and $t^{-1}(L)$.

- It turns out that the log pair groupoid is

  $$[\text{Pair}(M) : \text{Pair}(L)] = \text{Bl}_{\text{Pair}(L)}(\text{Pair}(M)) \setminus (p^{-1}(L \times M) \cup p^{-1}(M \times L))$$

  and the blow-down map

  $$\rho : [\text{Pair}(M) : \text{Pair}(L)] \to \text{Pair}(M)$$

  is a Lie groupoid morphism.
Examples of integrations

4. Elementary modification

Generalizing the log tangent algebroid, if $A$ is a Lie algebroid over $M$, and $B$ is a Lie subalgebroid over a hypersurface $L \subset M$, then we define the Lie algebroid $[A:B]$ with sheaf of sections

$$[A:B](U) = \{ X \in \Gamma(U, A) \mid X|_L \in \Gamma(U \cap L, B) \}.$$ 

That is, a section $X$ of $[A:B]$ is a section of $A$ such that when $X|_L$ is a section of $B$. 
Examples of integrations

4. Elementary modification

Generalizing the log tangent algebroid, if \( A \) is a Lie algebroid over \( M \), and \( B \) is a Lie subalgebroid over a hypersurface \( L \subset M \), then we define the Lie algebroid \([A:B]\) with sheaf of sections

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[A:B](U) = \{ X \in \Gamma(U, A) \mid X|_L \in \Gamma(U \cap L, B) \}.
\]

That is, a section \( X \) of \([A:B]\) is a section of \( A \) such that when \( X|_L \) is a section of \( B \).

- If \( \mathcal{G} \rightrightarrows M \) integrates \( A \) and \( \mathcal{H} \rightrightarrows L \) is a subgroupoid of \( \mathcal{G} \) integrating \( B \), then we may blow up \( \mathcal{G} \) along \( \mathcal{H} \), and remove the proper transforms of \( s^{-1}(L) \) and \( t^{-1}(L) \) as before:

\[
[\mathcal{G}:\mathcal{H}] = \text{Bl}_{\mathcal{H}}(\mathcal{G}) \setminus (p^{-1}(s^{-1}(L)) \cup p^{-1}(t^{-1}(L))).
\]
Examples of integrations

- it turns out that \([\mathcal{G} : \mathcal{H}]\) is a Lie groupoid integrating \([A : B]\) and the blow-down map

\[ p : [\mathcal{G} : \mathcal{H}] \to \text{Pair}(M) \]

is a Lie groupoid morphism.
Examples of integrations

- it turns out that $[G : H]$ is a Lie groupoid integrating $[A : B]$ and the blow-down map

$$\rho : [G : H] \to \text{Pair}(M)$$

is a Lie groupoid morphism.

Thank you!