Homework set # 5
Due on 2/20

0. The following problems from Artin “Algebra” edition 2: 15.8.1; 16.1.1 parts a,b,c

1. (1) Let \( \phi : F \rightarrow F' \) be an isomorphism of fields. Let \( f(x) \in F[x] \) be a polynomial and let \( f'(x) = \phi(f(x)) \) (here we are just applying \( \phi \) to the coefficients of \( f(x) \)). Let \( E \) be a splitting field for \( f(x) \) over \( F \) and let \( E' \) be a splitting field for \( f'(x) \) over \( F' \). Prove that the isomorphism \( \phi \) extends to an isomorphism \( \sigma : E \rightarrow E' \) (so in other words that \( \sigma \) restricted to \( F \) is just \( \phi \)). (Hint: first consider what happens when you adjoin one root of \( f(x) \) to \( F \) and one root of \( f'(x) \) to \( F' \), it might also be helpful to think of adjoining one root as a quotient of the polynomial ring).

(2) Using the first part, prove that any two splitting fields of a polynomial \( f(x) \in F[x] \) over a field \( F \) are isomorphic.

2. (1) For every non constant monic polynomial \( f \in F[x] \) where \( F \) is a field, let \( x_f \) denote a new variable in the polynomial ring \( R_f = F[\ldots, x_f, \ldots] \) (i.e. there will be infinitely many variables in this new polynomial ring). Now let \( I \) be the ideal in \( R_f \) generated by the polynomials \( f(x_f) \). Prove that \( I \) is a proper ideal (i.e. that \( I \neq R_f \)). (Hint: If it were proper then \( 1 \in I \) meaning that there would be a relation \( g_1 f_1(x_{f_1}) + \cdots g_n f_n(x_{f_n}) = 1 \) among finitely many of the \( f(x_f) \)'s. Now what would happen to this relation if you set each \( x_{f_i} \) equal to a root of \( f_i \) in some extension field of \( F \) and set the remaining variables showing up in the \( g_i \)'s to 0?)

(2) Observe that if \( I \) is not equal to \( R_f \) then \( I \) is contained in some maximal ideal \( M \) of \( R_f \). Prove that the field \( R_f/M \) contains a root of every non constant monic polynomial \( f \in F[x] \).

(3) Using the above work, prove that for any field \( F \) there exists an algebraically closed field \( K \) containing \( F \). (Hint: It might be useful to use the fact that a union of fields is a field (even a countable union)).