Multiple Choice

1.(6 pts) Find $\mathrm{d}z/\mathrm{d}t$ when $t = 0$, where $z = x^2 + y^2 + 2xy$, $x = \ln(t+1)$ and $y = e^{3t}$.

(a) 8 (b) 2 (c) 1

(d) 6 (e) 5

2.(6 pts) Calculate the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ at the point $(2, 4, 2)$ in the direction of the vector $\langle 1, 2, 1\rangle$.

(a) $\dfrac{24}{\sqrt{6}}$

(b) $\sqrt{6}$

(c) $-\dfrac{1}{12}\langle 1, 4, 1\rangle$

(d) -9.79

(e) $\left\langle \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}} \right\rangle$

3.(6 pts) What is the normal line to $z^2 = 9x^2 - 4y^2$ at $(2, 3, 0)$?

(a) $\langle 2, 3, 0\rangle + t\,\langle 36, -24, 0\rangle$

(b) $\langle 2, 3, 0\rangle + t\,\langle 18x, -8y, -2z\rangle$

(c) $\langle 2, 3\rangle + t\langle 36, -24\rangle$

(d) $36x - 24y = 0$

(e) $\langle 2, 3, 0\rangle + t\,\langle 24, 36, -12\rangle$

4.(6 pts) Find and classify the critical points of $f(x, y) = x^2 + 6xy - 4y^2$.

(a) $(0, 0)$, saddle point.

(b) $(0, 0)$, local maximum.

(c) $(0, 0)$, local minimum.

(d) $(1, 1)$, saddle point.

(e) $(1, 1)$, local maximum.

5.(6 pts) Suppose $f(x,y) = x^2y$ with domain $D = \{(x,y)|x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$. What is the absolute maximum value of $f(x,y)$?

(a) 2

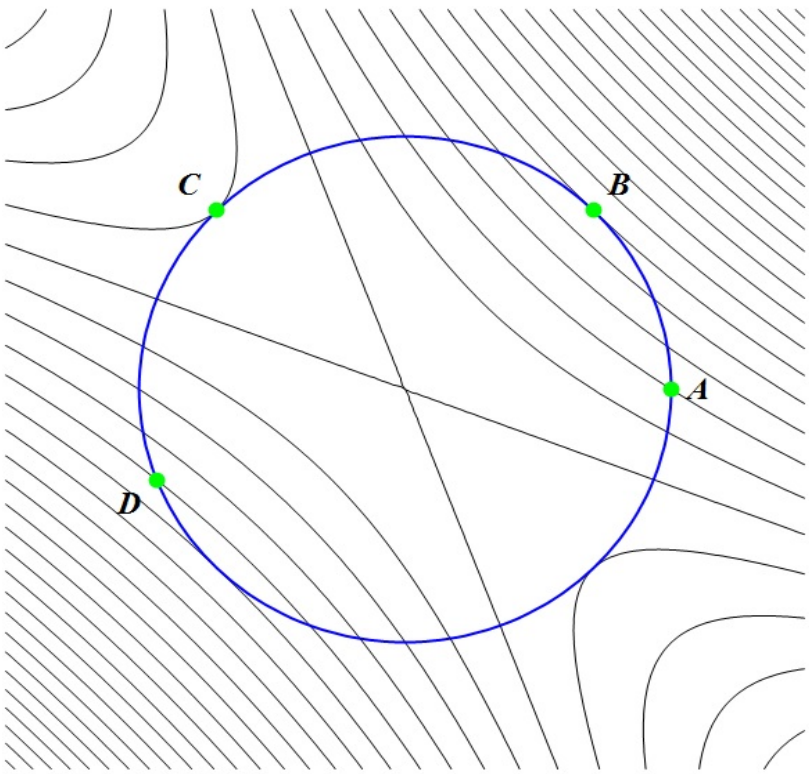
(b) 1

(c) 3

(d) 4

(e) 5

6.(6 pts) Consider the following contour plot for a function $f(x, y)$:



The circle is a level curve $g(x, y) = k$. Which of the following must ALWAYS be true?

(a) Subject to $g(x, y) = k$, $f(x, y)$ has a possible extremum at C.

(b) Subject to $g(x, y) = k$, $f(x, y)$ has a possible maximum at A.

(c) Subject to $g(x, y) = k$, $f(x, y)$ has a possible minimum at D.

(d) Subject to $g(x, y) = k$, $f(x, y)$ has an absolute maximum at B.

(e) $f(x, y)$ has a possible absolute maximum or absolute minimum at C.

7.(6 pts) Evaluate the following double integral

$$\iint_R (5 - x)$$

for $R = \{(x, y) \,|\, 0 \leq x \leq 4, 0 \leq y \leq 3\}$.

(a) 36

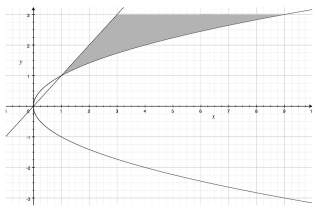
(b) 24

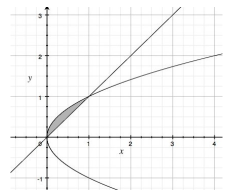
(c) 60

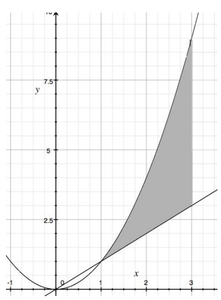
(d) 12

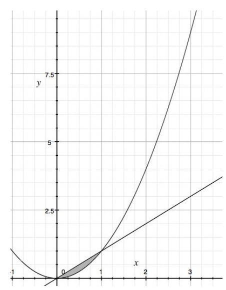
(e) 52

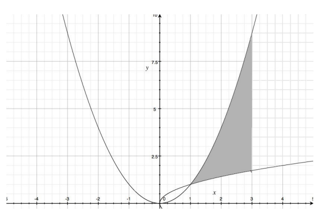
8.(6 pts) Consider the double integral of a function f over a region R, $\iint_R f$. Suppose $\iint_R f = \int_1^3 \int_y^{y^2} f(x,y)\,dx\,dy$. Which gray region below is R?

(a) 

(b) 

(c) 

(d) 

(e) 

Partial Credit

You must show your work on the partial credit problems to receive credit!

9.(12 pts.) (a) Find an equation for the tangent line (in vector or parametric form) at the point $(2, 2, 1)$ to the curve of intersection of the two surfaces $g(x, y, z) = 2x^2+2y^2+z^2 = 17$ and $h(x, y, z) = x^2 + y^2 - 3z^2 = 5$. (8 pts)

(b) Suppose $f(x, y, z)$ is a function with $\nabla f = \langle 0, 1, 0 \rangle$ at the point $(2, 2, 1)$. Starting at $(2, 2, 1)$, which direction should one travel along the curve of intersection in order to increase f? (2 pts)

Note: You can specify a direction along the curve by saying whether the variable in your equation from (a) would increase or decrease, or by choosing a vector tangent to the curve.

10.(12 pts.) Find the absolute maximum and minimum of $f(x, y, z) = 2x+y$ with respect to the constraints $g(x, y, z) = 2x^2 + z^2 = 4$ and $h(x, y, z) = 2x + y + 3z = 6$.

11.(12 pts.) Find and classify all critical points of $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

12.(12 pts.) A cylinder containing an incompressible fluid is being squeezed from both ends. If the length of the cylinder is changing at a rate of -3m/s, calculate the rate at which the radius is changing when the radius is 2m and the length is 1m. (Note: An incompressible fluid is a fluid whose volume does not change.)

13.(12 pts.) Evaluate $\iint_R 4xy$ where R is the region bounded above by $y = \sqrt{x}$ and below by $y = x^3$.

Name: ____________________________

Instructor: ANSWERS

Math 20550, Exam 2
January 1, 01

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name is on every page in case pages become detached
- Be sure that you have all 8 pages of the test.
- Each multiple choice question is 6 points, each partial credit problem is 12 points. You will receive 4 extra points.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3.	(•)	(b)	(c)	(d)	(e)
4.	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	
9.	
10.	
11.	
12.	
13.	
Extra Points.	4
Total	

9.(a) The line is in the tangent plane to each surface, so its direction is perpendicular to both normal vectors. The normal vectors are $\nabla g = \langle 4x, 4y, 2z\rangle = \langle 8, 8, 2\rangle$ and $\nabla h = \langle 2x, 2y, -6z\rangle =< 4, 4, -6 >$. The cross product $\nabla g \times \nabla h = \langle -56, 56, 0\rangle$ will serve as a direction vector. $\langle 2, 2, 1\rangle + t\,\langle -56, 56, 0\rangle$ is an equation for the tangent line.

(b) Let $\mathbf{u}$ be a unit vector which points in the same direction as $\langle -56, 56, 0\rangle$. Since $D_{\mathbf{u}}f = \dfrac{\langle 0, 1, 0\rangle \cdot \langle -56, 56, 0\rangle}{56\sqrt{2}} = \dfrac{1}{\sqrt{2}} > 0$ at $(2, 2, 1)$, one should increase t in order to increase f.

10.

$$\nabla f = \langle 2, 1, 0\rangle$$
$$\nabla g = \langle 4x, 0, 2z\rangle$$
$$\nabla h = \langle 2, 1, 3\rangle$$

So

$$2 = 4x\lambda + 2\mu$$
$$1 = \mu$$
$$0 = 2\lambda + 3\mu$$

Using the second equation on the first equation we get

$$2 = 4x\lambda + 2$$

This reduces to

$$0 = 4x\lambda$$

This implies $x = 0$ or $\lambda = 0$. Let try $\lambda = 0$ in the third equation above. That yields $0 = 3$ which is a contradiction. So $x = 0$. Now we can use our restraints to find y, z. Using $g(x, y, z) = 2x^2+z^2 = 4$, we get $z^2 = 4$ or $z = \pm 2$. Using $h(x, y, z) = 2x+y+3z = 6$ we see that when $x = 0$ and $z = 2$ that $y + 6 = 6$ so that $y = 0$ and that when $x = 0$ and $z = -2$ that $y - 6 = 6$ so that $y = 12$. So our critical points are $(0, 0, 2)$ and $(0, 12, 2)$. $f(0, 0, 2) = 0$ for an absolute minimum and $f(0, 12, -2) = 12$ for an absolute maximum.

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11. Begin by finding all first and second partial derivatives: $f_x = 6xy - 6x$, $f_y = 3x^2 + 3y^2 - 6y$, $f_{xx} = 6y - 6$, $f_{xy} = 6x$, $f_{yy} = 6y - 6$. We now need the critical points. Find these by solving the equations

$$f_x = 6xy - 6x = 0$$
$$f_y = 3x^2 + 3y^2 - 6y = 0$$

The first equation factors as $6x(y - 1) = 0$ so it will be zero if $x = 0$ or $y = 1$. The most common mistake here was to forget the $x = 0$ solution. To find the critical points we can plug these values into f_y and solve for the remaining variable. For $x = 0$ we have $f_y = 3y^2 - 6y = 0$ which implies $y = 0$ or $y = 2$. For $y = 1$ we have $f_y = 3x^2 - 3 = 0$ which implies $x = 1$ or $x = -1$. So if $x = 0$ we have the critical points $(0, 0)$ and $(0, 2)$.

If $y = 1$ we have the critical points $(1, 1)$ and $(-1, 1)$. Now all we need to do is classify the critical points. The discriminant $D(x, y)$ is given by

$$D(x, y) = (6y - 6)^2 - 36x^2$$

$(0, 0)$: $D(0, 0) = 36 > 0$ and $f_{xx}(0, 0) = -6 < 0$. $(0, 2)$: $D(0, 2) = 36 > 0$ and $f_{xx}(0, 2) = 6 > 0$. $(1, 1)$: $D(1, 1) = -36 < 0$. $(-1, 1)$: $D(-1, 1) = -36 < 0$. So $(0, 0)$ is a relative max, $(0, 2)$ is a relative min, and $(1, 1), (-1, 1)$ are saddle points.

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12. We have $V = \pi r^2 \ell$, where V is the volume, r the radius and ℓ the length, and each of r and ℓ are functions of the time, t. Since the fluid is incompressible $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$.

By the chain rule, this is

$$0 = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial V}{\partial \ell}\frac{\mathrm{d}\ell}{\mathrm{d}t} = 2\pi r\ell\frac{\mathrm{d}r}{\mathrm{d}t} + \pi r^2\frac{\mathrm{d}\ell}{\mathrm{d}t}.$$

Filling in $\mathrm{d}\ell/\mathrm{d}t = -3$ we obtain $\mathrm{d}r/\mathrm{d}t = 3\pi r^2/2\pi r\ell = 3r/2\ell$. When $r = 2$, $\ell = 1$ this gives $\mathrm{d}r/\mathrm{d}t = 3\mathrm{m/s}$.

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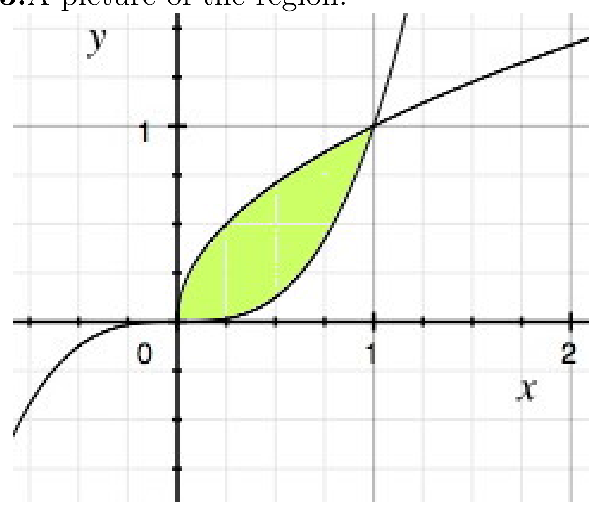
13. A picture of the region:



To set up the double integral as an iterated integral $dy\,dx$ we first need bounds for x. Clearly we start when $x = 0$ and end when $x = 1$ or more formally we need to find when $x^3 = \sqrt{x}$ or $x^6 = x$ or $x = 0$, $x^5 = 1$, which has solutions $x = 0,\ 1$. Then the limits on the inner integral are x^3 at the bottom and $\sqrt{x}$ at the top.

$\iint_R 4xy = \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy\,dy\,dx = \int_0^1 2xy^2\Big|_{x^3}^{\sqrt{x}} dx = \int_0^1 2x^2 - 2x^7\,dx = \left.\frac{2x^3}{3} - \frac{2x^8}{8}\right|_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$

Or we could set up the double integral as an iterated integral $dx\,dy$. This time we need to know the y-coordinates of the intersection points but the same algebra as above gives $y = 0,\ 1$. The limits on the inner integral start at the right hand curve whose

x-coordinate in terms of y is $x = y^2$. The upper limit is the x-coordinate of the right-han curve in terms of y which is $x = \sqrt[3]{y}$. Hence

$\iint_R 4xy = \int_0^1 \int_{y^2}^{\sqrt[3]{y}} 4xy \, dx \, dy = \int_0^1 2x^2y\Big|_{x=y^2}^{x=\sqrt[3]{y}} dx = \int_0^1 2y^{5/3} - 2y^5 \, dy = 2\frac{3}{8}y^{8/3} - \frac{2y^6}{6}\Big|_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$.