Answer Key 1

MATH 20550: Calculus III Name:______________________

Exam III *November 15, 2011* Instructor and Section:______________________

As a member of the Notre Dame community, I will not participate in or tolerate academic dishonesty.

Please sign ____

Record your answers to the multiple choice problems by placing an $\times$ through one letter for each problem on this page. There are 10 multiple choice questions worth 6 points each and 3 partial credits problems worth 14 points each. On the partial credit problems try to simplify your answer and indicate your final answer clearly. *You must show your work and all important steps to receive credit.*

You may use a calculator if you wish.

Score

DO NOT WRITE IN THIS	COLUMN
Multichoice	
10	
11	
12	
Total	

1. • b c d e
2. a • c d e
3. a • c d e
4. a • c d e
5. a b c d •
6. • b c d e
7. • b c d e
8. a • c d e
9. a b • d e
10. a • c d e

Practice Exam 3.3
Solutions to Partial Credit Problems

November 6, 2012

11. The cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 2$ (radius $\sqrt{2}$) intersect at

$$z^2 + z^2 = 2 \overset{z\geq 0}{\Rightarrow} z = 1 \Rightarrow \sqrt{2}\cos\phi = 1 \Rightarrow \phi = \frac{\pi}{4}.$$

Thus, the solid region E above the cone and below the sphere is:

$$E = \left\{(r,\theta,\phi) : 0 \leq r \leq \sqrt{2},\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{4}\right\}.$$

The mass of the solid is equal to:

$$\begin{aligned} m &= \iiint_E \rho(x,y,z)dV \\ &= \int_0^{\sqrt{2}} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} k\ r^2 \sin\phi\ d\phi\ d\theta\ dr \\ &= 2\pi\ k \left[\frac{r^3}{3}\right]_{r=0}^{r=\sqrt{2}} \left[-\cos\phi\right]_{\phi=0}^{\phi=\frac{\pi}{4}} \\ &= 2\pi\ k\ \frac{2\sqrt{2}}{3}\ \frac{\sqrt{2}-1}{\sqrt{2}} \\ &= k\ \frac{4\pi}{3}(\sqrt{2}-1). \end{aligned}$$

The z-component of the centre of mass is:

$$\begin{aligned}
\bar{z} &= \frac{1}{m}\iiint_E z\rho(x,y,z)dV \\
&= \frac{1}{m}\int_0^{\sqrt{2}}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}(r\cos\phi)\ k\ r^2\sin\phi\ d\phi\ d\theta\ dr \\
&= \frac{1}{m}\ k\int_0^{\sqrt{2}}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} r^3\ \ \cos\phi\sin\phi\ d\phi\ d\theta\ dr \\
&= \frac{2\pi}{m}\ k\left[\frac{r^4}{4}\right]_{r=0}^{r=\sqrt{2}}\ \frac{1}{2}\left[\sin^2\phi\right]_{\phi=0}^{\phi=\frac{\pi}{4}} \\
&= \frac{\pi}{2m}\ k \\
&= \frac{3}{8(\sqrt{2}-1)}.
\end{aligned}$$

12. The four sides of the given square R are given by:

$$y = x, \quad (1,1)\to(2,2)$$

$$\begin{aligned}
y = -x+4, &\quad (2,2)\to(1,3) \\
y = x+2, &\quad (1,3)\to(0,2) \\
y = -x+2, &\quad (0,2)\to(1,1).
\end{aligned}$$

Under the change of variables $u = x-y,\ v = x+y$, the four sides of the square respectively become:

$$u = 0,\ 1\le v\le 4 \quad (0,2)\to(0,4)$$

$$\begin{aligned}
v = 4,\ -2\le u\le 0 &\quad (0,4)\to(-2,4) \\
u = -2,\ 2\le v\le 4 &\quad (-2,4)\to(-2,2) \\
v = 0,\ -2\le u\le 0 &\quad (-2,2)\to(0,2).
\end{aligned}$$

Also, since

$$x = \frac{1}{2}(u+v), \quad y = \frac{1}{2}(v-u)$$

the Jacobian of the transformation is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

Hence,

$$\iint_R \frac{x-y}{x+y} dA = \int_{-2}^{0} \int_{2}^{4} \frac{u}{v} \left| \frac{1}{2} \right| dv du = \frac{1}{2} \left[\frac{u^2}{2} \right]_{u=-2}^{u=0} [\ln v]_{v=2}^{v=4} = -\ln 2.$$

13. Green's theorem for the vector field $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ and with L denoting the positively oriented (anti-clockwise) boundary of a region D:

$$\int_L \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y) dA.$$

We have:

$$P(x,y) = x^2 y, \quad Q(x,y) = -xy^2,$$

hence

$$P_y = x^2, \quad Q_x = -y^2.$$

Thus, noting that the contour C specified by the problem has **negative** orientation (goes in the **clockwise** direction), we have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\iint_D \left(-y^2 - x^2\right) dA = \iint_D \left(x^2 + y^2\right) dA.$$

The region D enclosed between the lower semicircle of radius 1 and the x-axis is

$$D = \{(r, \theta) : 0 \le r \le 1,\ \pi \le \theta \le 2\pi\},$$

therefore using plane polars:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \int_\pi^{2\pi} (r^2)\ r\ dr d\theta = \pi \left[\frac{r^4}{4} \right]_{r=0}^{r=1} = \frac{\pi}{4}.$$