MATH 20550: Calculus III Name: Solutions

Final *December 11, 2012* Instructor and Section:

As a member of the Notre Dame community, I will not participate in or tolerate academic dishonesty.

Please sign

Record your answers to the multiple choice problems by placing an × through one letter for each problem on this page. There are 20 multiple choice questions worth 7 points each. You start with 10 points for a total of 150 points.

You may use a calculator if you wish.

1.	a	b	c	d	~~e~~	11.	a	b	~~c~~	d	e
2.	~~a~~	b	c	d	e	12.	a	b	~~c~~	d	e
3.	a	b	c	~~d~~	e	13.	a	b	c	d	~~e~~
4.	a	~~b~~	c	d	e	14.	a	~~b~~	c	d	e
5.	a	b	~~c~~	d	e	15.	a	~~b~~	c	d	e
6.	a	b	c	~~d~~	e	16.	a	~~b~~	c	d	e
7.	a	~~b~~	c	d	e	17.	a	b	c	~~d~~	e
8.	a	~~b~~	c	d	e	18.	a	b	~~c~~	d	e
9.	a	b	c	d	~~e~~	19.	a	b	~~c~~	d	e
10.	a	b	c	~~d~~	e	20.	a	b	c	d	~~e~~

1. A spaceship travels through space with acceleration, $\mathbf{a}(t) = 2\mathbf{j} - \mathbf{k}$, its initial velocity is $\mathbf{v}(0) = \langle 1, -2, 0\rangle$, and its initial position is $\mathbf{r}(0) = \langle 2, -1, 3\rangle$. What is its position at time $t = 2$?

(a) $(4, 4, -2)$ (b) $(2, 0, -2)$ (c) $(4, 3, 1)$ (d) $(2, 4, -2)$ (e) $(4, -1, 1)$

$\vec{v}(t) = \langle C_1, 2t + C_2, -t + C_3\rangle = \langle 1, 2t-2, -t\rangle$

$\vec{r}(t) = \langle t + C_1, t^2 - 2t + C_2, -\frac{1}{2}t^2 + C_3\rangle = \langle t+2, t^2-2t-1, -\frac{1}{2}t^2+3\rangle$

$\vec{r}(2) = \langle 4, -1, 1\rangle$

2. Joe the baker is back; this time he's making solid ice cream cones. He is shaping a cone of height 4 cm and radius 3 cm. Presently, the height is increasing at a rate of 4 cm/s. Assuming that the volume of the cone is constant, how fast is the radius changing? Note: The volume of a cone is $V = \dfrac{1}{3}\pi r^2 h$.

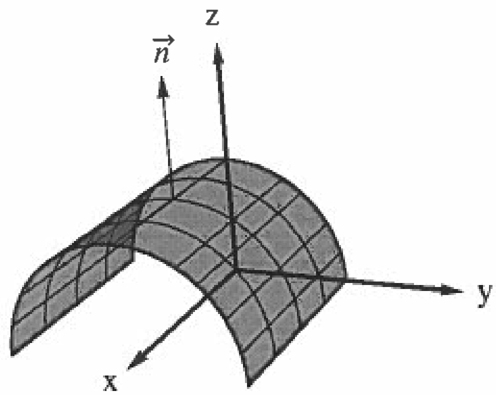
(a) $-\dfrac{3}{2}$ cm/s (b) $-\dfrac{5}{2}$ cm/s (c) $-\dfrac{1}{2}$cm/s (d) -1 cm/s (e) -2 cm/s

$$0 = \frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt} = \frac{2}{\cancel{3}}\pi (\cancel{3})(4)\frac{dr}{dt} + \frac{1}{\cancel{3}}\pi (3)^{\cancel{2}}(4)$$

$$= 8\pi \frac{dr}{dt} + 12\pi$$

$$\frac{dr}{dt} = \frac{-12\pi}{8\pi} = \frac{-3}{2}$$

3. Find the flux of the field $\mathbf{F} = \langle z^3, z, -y \rangle$ through the surface $z = \sqrt{1-y^2}$ with $0 \le x \le 2$ and upward normal (see graph below).



(a) 8 (b) 10 (c) −5 (d) 0 (e) 6

$\vec{r}(x,y) = \langle x, y, \sqrt{1-y^2} \rangle$, $0 \le x \le 2$, $-1 \le y \le 1$

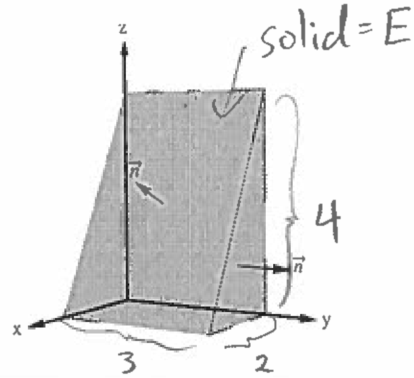
$\vec{r}_x = \langle 1,0,0 \rangle$ $\vec{r}_x \times \vec{r}_y = \langle 0, \frac{y}{\sqrt{1-y^2}}, 1 \rangle$

$\vec{r}_y = \langle 0, 1, \frac{-y}{\sqrt{1-y^2}} \rangle$

$\vec{F}(\vec{r}(x,y)) = \langle (1-y^2)^{3/2}, (1-y^2)^{1/2}, -y \rangle$

$\vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y) = 0 + y - y = 0$

4. Use the Divergence Theorem to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = (x^2 - 2x)\mathbf{i} + (y\sin(z) - 2xy)\mathbf{j} + (\cos(z))\mathbf{k}$ and S is the surface of the solid enclosed by the following planes $z = 4 - 2x$, $z = 0$, $x = 2$, $x = 0$, $y = 0$, and $y = 3$ with the outward normal (see graph below).



(a) 36 (b) −24 (c) 0 (d) −12 (e) 18

$$\operatorname{div}\vec{F} = (\cancel{2x}-2)+(\cancel{\sin z}-\cancel{2x})+(\cancel{-\sin z}) = -2$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\operatorname{div}\vec{F})\,dV = \iiint_E -2\,dV = -2\cdot \mathrm{vol}(E) = -2\left(\frac{2\cdot 3\cdot 4}{2}\right)$$

$$= -24$$

5. Find the absolute maximum value of $f(x,y) = x^3 - 3x - y^4 + 8y^2$ on the square region $D = \{(x,y) \mid 0 \le x \le 2, 1 \le y \le 3\}$.

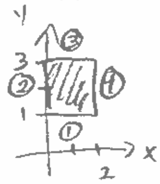
Hints: $f(0,1) = 7$, $f(0,3) = -9$, $f(2,1) = 9$, $f(2,3) = -7$, and $f(0,2) = 16$.

(a) 20 (b) 22 (c) 18 (d) 16 (e) 17

$\nabla f = \langle 3x^2-3, -4y^3+16y \rangle$ $(1,2)$ inside D

$x = \pm 1$ $y=0, y=\pm 2$

$x \ne -1$ $y \ne 0, -2$ } not in D



① $y=1$: $f(x,1) = x^3-3x+7 = g_1(x)$. $g_1'(x) = 3x^2-3$, $x=1$, $(1,1)$

② $x=0$: $f(0,y) = -y^4+8y^2 = g_2(y)$; $g_2'(y) = -4y^3+16y$, $y=2$, $(0,2)$

③ $y=3$: $f(x,3) = x^3-3x-9 = g_3(x)$, $g_3'(y) = 3x^2-3$, $x=1$ $(1,3)$

④ $x=2$: $f(2,y) = 2-y^4+8y^2 = g_4(y)$, $g_4'(y) = -4y^3+16y$, $y=2$, $(2,2)$

	Points	val
	$(1,2)$	$1-3-16+32=16$
	$(1,1)$	$1-3-1+8=5$
	$(0,2)$	$-16+32=16$
	$(1,3)$	$1-3-81+72=-11$
	$(2,2)$	$8-6-16+32=18$
corners of D	$(0,1)$	7
	$(0,3)$	-9
	$(2,3)$	$8-6-81+72=-7$
	$(2,1)$	$8-6-1+8=9$

6. For the function $f(x,y,z) = x + 2y$ what is the maximum value subject to the constraints $\underbrace{x+y+z}_{g} = 1$ and $\underbrace{y^2+z^2}_{h} = 8$?

(a) -3 (b) 0 (c) 6 (d) 5 (e) 8

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 1 \\ h = 8 \end{cases} \Rightarrow \begin{cases} 1 = \lambda & ① \\ 2 = \lambda + 2\mu y & ② \Rightarrow 1 = 2\mu y \\ 0 = \lambda + 2\mu z & ③ \Rightarrow -1 = 2\mu z \\ x+y+z = 1 & ④ \\ y^2+z^2 = 8 & ⑤ \end{cases}$$

$\Rightarrow 2\mu y = -2\mu z$

$\Rightarrow \mu = 0$ or $y = -z$

$\mu = 0$ contradicts ①&②&③ $\Rightarrow \mu \ne 0$

$\boxed{y=-z}$ ⑤: $2y^2 = 8 \Rightarrow y = \pm 2$

$\boxed{y=2} \Rightarrow z=-2$ ④ $\Rightarrow x=1$

$\boxed{y=-2} \Rightarrow z=2$ ④ $\Rightarrow x=1$

pt	$(1,2,-2)$	$(1,-2,2)$
val	5	-3

7. Let $\mathbf{r}(t) = \langle 2t, t^2, t^3 \rangle$. Compute the osculating plane at $t = 1$.

(a) $6x + 12y + 4z = 4$ (b) $6x - 12y + 4z = 4$ (c) $2y + z = 2$

(d) $x + 2y + 3z = 7$ (e) $6x - 12y + 4z = 0$

Use $\vec{r}\,'(1) \times \vec{r}\,''(1) \parallel \vec{B}(1)$ for perpendicular vector

$\vec{r}\,'(t) = \langle 2, 2t, 3t^2 \rangle$, $\vec{r}\,'(1) = \langle 2, 2, 3 \rangle$ $\vec{r}\,'(1) \times \vec{r}\,''(1) = \langle 6, -12, 4 \rangle$

$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$, $\vec{r}\,''(1) = \langle 0, 2, 6 \rangle$

$\vec{r}(1) = \langle 2, 1, 1 \rangle$ $\langle 6, -12, 4 \rangle \cdot \langle x-2, y-1, z-1 \rangle = 6x \cancel{-12} - 12y \cancel{+12} + 4z - 4 = 0$

$6x - 12y + 4z = 4$

8. Compute the line integral

$$\int_C \sqrt{1 - y^2}\, ds,$$

where C is the helix given by $x = \cos t$, $y = \sin t$, $z = \sqrt{3}t$. $0 \le t \le \dfrac{\pi}{2}$.

(a) $2\sqrt{3}$ (b) 2 (c) 1 (d) $\dfrac{7}{12}$ (e) 0

$\vec{r}\,'(t) = \langle -\sin t, \cos t, \sqrt{3} \rangle$ $|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 3} = \sqrt{4} = 2$

$\cos t \geq 0$ on $0 \le t \le \pi/2$

$$\int_C \sqrt{1 - y^2}\, ds = \int_0^{\pi/2} \sqrt{1 - \sin^2 t}\, (2\,dt) = \int_0^{\pi/2} 2\sqrt{\cos^2 t}\, dt = \int_0^{\pi/2} 2\cos t\, dt$$

$$= 2\sin t \Big|_0^{\pi/2} = 2$$

9. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle x^2+y^2, y^2\rangle$ and C is the circular arc given by the parametrization $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, where $0 \le t \le \pi$.

(a) 0 (b) 3 (c) 4 (d) -1 (e) -2

$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = \langle 1, \sin^2 t\rangle \cdot \langle -\sin t, \cos t\rangle = -\sin t + \sin^2 t\cos t$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} (-\sin t + \sin^2 t\cos t)\,dt = \left(\cos t + \frac{1}{3}\sin^3 t\right)\Big|_0^{\pi} = -1-(1) = -2$

10. Use the Fundamental Theorem of Line Integrals to compute the line integral

$$\int_C \langle \underset{P}{3x^2yz}, \underset{Q}{x^3z + ze^{yz}}, \underset{R}{x^3y + ye^{yz} + \cos z}\rangle \cdot d\mathbf{r}, \quad = \int_C \nabla f \cdot d\vec{r}$$

where C is any smooth curve in $\mathbb{R}^3$ from $(2,2,0)$ to $(5,0,\pi/2)$.

(a) -1 (b) 0

(c) 2 (d) 1

(e) the answer depends on the choice of C

$f = \int P\,dx = x^3yz + g(y,z) = x^3yz + e^{yz} + h(z)$

$f_y = x^3z + g_y(y,z) = Q = x^3z + ze^{yz}$

$\Rightarrow g_y(y,z) = ze^{yz} \Rightarrow g(y,z) = e^{yz} + h(z)$

$f_z = x^3y + ye^{yz} + h'(z) = R = x^3y + ye^{yz} + \cos z$

$\Rightarrow h'(z) = \cos z \Rightarrow h(z) = \sin z + C$

$(C=0)$

Potential: $f = x^3yz + e^{yz} + \sin z$

$\int_C \nabla f\cdot d\vec{r} = f(5,0,\pi/2) - f(2,2,0) = (0 + \cancel{e^0} + 1) - (0 + e^0 + 0) = 1$

11. Consider a hemispherical solid bounded below by the xy-plane and above by the hemisphere given by $z = \sqrt{1-x^2-y^2}$. Suppose that the density of the solid is constant. The volume of this hemisphere is $\frac{2\pi}{3}$. Find the z coordinate of center of mass, $\bar{z}$.

(a) 1 (b) 0 (c) $\frac{3}{8}$ (d) $\frac{1}{3}$ (e) $\frac{1}{2}$

density $= \delta$

$$\bar{z} = \frac{\iiint_E \delta z\,dV}{\iiint_E \delta\,dV} = \frac{\iiint_E z\,dV}{\frac{2\pi}{3}} = \frac{3}{2\pi}\int_0^1\int_0^{\frac{\pi}{2}}\int_0^{2\pi} \rho^3 \sin\varphi\cos\varphi\,d\theta\,d\varphi\,d\rho$$

$$= 3\int_0^1\int_0^{\pi/2} \rho^3 \sin\varphi\cos\varphi\,d\varphi\,d\rho = \frac{3}{2}\int_0^1 \rho^3 \sin^2\varphi\Big|_0^{\pi/2} d\rho = \frac{3}{2}\int_0^1 \rho^3\,d\rho = \frac{3}{8}$$

12. Find the rate of change of $z = y^2e^{x+y}$ in the direction $\langle -1, 1\rangle$ at the point $(0,1)$:

$\sqrt{2}$

(a) -1 (b) 0 (c) $\sqrt{2}e$ (d) $2e$ (e) $\langle -1, e\rangle$

$$\nabla z = \langle y^2e^{x+y},\ 2ye^{x+y} + y^2e^{x+y}\rangle$$

$$\nabla z(0,1) = \langle e, 3e\rangle$$

$$\langle e, 3e\rangle \cdot \frac{1}{\sqrt{2}}\langle -1, 1\rangle = \frac{1}{\sqrt{2}}(-e+3e) = \sqrt{2}e$$

Skip?

13. Assume that all people can safely handle a normal component of acceleration, a_N, of 50 (meters per second squared) for a few seconds in a curve. To fit the roller coaster into the given space the engineers need to build a curve with curvature 1/8 (inverse meters). How fast can the roller coaster go through that curve safely?

(a) 150 (meters per second) (b) 90 (meters per second) (c) 10 (meters per second)

(d) 30 (meters per second) (e) 20 (meters per second)

$a_N = K v^2$ $\Leftarrow$ $50 = \frac{1}{8} v^2 \Rightarrow v^2 = 400 \Rightarrow v = 20$

curvature speed

14. Determine which iterated integral gives the values of the surface integral $\iint_S \sqrt{1 + 4x^2 + 4y^2}\, dS$, where S is the surface defined by $z = f(x, y) = x^2 + y^2$ for $0 \le z \le 4$.

(a) $\int_0^{2\pi} \int_0^2 (1 + 4r^2) dr d\theta$ (b) $\int_0^{2\pi} \int_0^2 (1 + 4r^2) r dr d\theta$ (c) $\int_0^{2\pi} \int_0^2 \sqrt{(1 + 4r^2)} r dr d\theta$

(d) ~~$\int_0^{2\pi} \int_0^4 (1 + 4r^2) r dr d\theta$~~ (e) ~~$\int_0^{2\pi} \int_0^4 (1 + 4r^2) dr d\theta$~~

$z = x^2 + y^2,\ 0 \le z \le 4$ sits over $x^2 + y^2 \le 4$ in xy-plane

$\sqrt{1+4x^2+4y^2} = \sqrt{1+4r^2}$ $dS = |\vec{r}_x \times \vec{r}_y|\, dA = \sqrt{4x^2+4y^2+1}\, dA$

$= \sqrt{1+4r^2}\, r dr d\theta$

Parametrization: $\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$

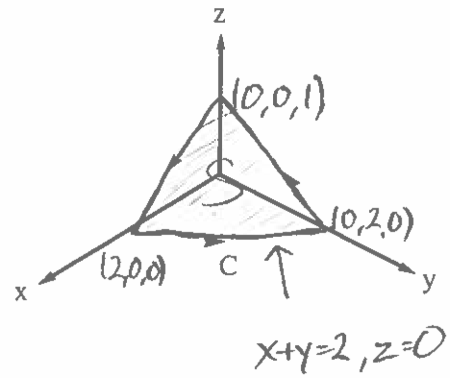
$\vec{r}_x = \langle 1, 0, 2x \rangle$

$\vec{r}_y = \langle 0, 1, 2y \rangle$

$\vec{r}_x \times \vec{r}_y = \langle -2x, -2y, 1 \rangle$

$\iint_S \sqrt{1+4x^2+4y^2}\, dS = \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2} \left(\sqrt{1+4r^2}\, r dr d\theta \right)$

15. A particle travels under the effect of the force field $\mathbf{F} = \langle e^{\sin x^2}, z, 3y \rangle$ on a path C which consists of straight line segments, starting from the point (2,0,0) and moving to (0,2,0), then to (0,0,1) and then back to (2,0,0). (Hint: C is the intersection of the plane $x + y + 2z = 2$ with the xy, xz, and yz coordinate planes.) Use Stokes' Theorem to compute the work done along C, $\int_C \mathbf{F} \cdot d\mathbf{r}$.

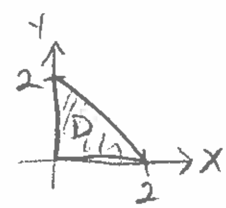


upward orientation

(a) 3 (b) 2 (c) 0 (d) 5 (e) 1

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ e^{\sin x^2} & z & 3y \end{vmatrix} = \langle 3-1, 0, 0 \rangle = \langle 2, 0, 0 \rangle$$

Parametrize surface S: $\vec{r}(x,y) = \langle x, y, \frac{2-x-y}{2} \rangle$,



$\vec{r}_x = \langle 1, 0, -\frac{1}{2} \rangle$ $\quad \vec{r}_x \times \vec{r}_y = \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle$

$\vec{r}_y = \langle 0, 1, -\frac{1}{2} \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot d\vec{S} = \iint_D \langle 2, 0, 0 \rangle \cdot \langle \tfrac{1}{2}, \tfrac{1}{2}, 1 \rangle \, dA$$

$$= \iint_D dA = \text{area}(D) = \tfrac{1}{2}(2)(2) = 2$$

16. The vector projection of the vector $\mathbf{b} = \langle 3, -1, 1 \rangle$ onto the vector $\mathbf{a} = \langle 2, 1, -2 \rangle$ is

(a) $\langle 2, 1, -2 \rangle$ (b) $\frac{1}{3}\langle 2, 1, -2 \rangle$ (c) $\frac{2}{3}\langle 2, 1, -2 \rangle$ (d) $\frac{1}{3}\langle 1, 1, 1 \rangle$ (e) $\frac{1}{3}\langle 3, -1, 1 \rangle$

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\vec{a} = \frac{6-1-2}{4+1+4}\langle 2,1,-2\rangle = \frac{3}{9}\langle 2,1,-2\rangle$$

17. Suppose a particle's position is given by $\mathbf{r}(t) = \langle 2t, t^2, t^3 \rangle$. What is the tangential component of the acceleration a_T when $t = 1$?

(a) $\sqrt{17}$ (b) $\frac{9}{\sqrt{6}}$ (c) 22 (d) $\frac{22}{\sqrt{17}}$ (e) 0

$$a_T = \frac{\vec{r}\,''(t)\cdot\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

$\vec{r}\,'(t) = \langle 2, 2t, 3t^2 \rangle$, $\vec{r}\,'(1) = \langle 2, 2, 3 \rangle$

$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$, $\vec{r}\,''(1) = \langle 0, 2, 6 \rangle$

$$a_T = \frac{0+4+18}{\sqrt{4+4+9}} = \frac{22}{\sqrt{17}}$$

18. Find the tangent plane to the surface $\mathbf{r}(u, v) = \langle u + 2v + 2, u^2v + v + 1, v^3 + 2u - 1\rangle$ when $u = v = 0$.

(a) $2x + y - 2z = -9$

(b) $2x + y - 5 = 0$

(c) $-2x + 4y + z = -1$

(d) $x + 2z = 1$

(e) $-2x + 4y + z = 2$

$\vec{r}_u = \langle 1, 2uv, 2\rangle$ $\quad \vec{r}_u(0,0) = \langle 1, 0, 2\rangle$

$\vec{r}_v = \langle 2, u^2+1, 3v^2\rangle$ $\quad \vec{r}_v(0,0) = \langle 2, 1, 0\rangle$

$\vec{n} = \vec{r}_u(0,0) \times \vec{r}_v(0,0) = \langle -2, 4, 1\rangle$

$\vec{r}(0,0) = \langle 2, 1, -1\rangle$

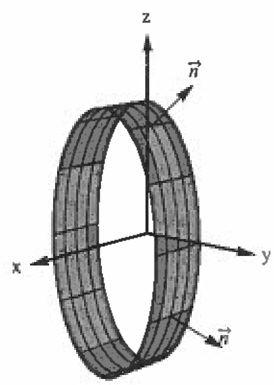
$0 = \langle -2, 4, 1\rangle \cdot \langle x-2, y-1, z+1\rangle = -2x + \cancel{4} + 4y - \cancel{4} + z + 1$

$-2x + 4y + z = -1$

19. Compute the flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

for the vector field $\mathbf{F} = \langle e^x, xy, xz\rangle$ across the open ended cylinder $4y^2 + z^2 = 4$ from $x = 0$ to $x = 1$ with outward normal (see graph below).



(a) $-\pi$ (b) 1 (c) 2π (d) π (e) 0

Parametrize : $\vec{r}(\theta, x) = \langle x, \cos\theta, 2\sin\theta\rangle$, $0 \le x \le 1$, $0 \le \theta \le 2\pi$

$\vec{r}_\theta = \langle 0, -\sin\theta, 2\cos\theta\rangle$

$\vec{r}_x = \langle 1, 0, 0\rangle$

$\vec{r}_\theta \times \vec{r}_x = \langle 0, 2\cos\theta, \sin\theta\rangle$

right order ✓

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^{2\pi} \langle e^x, x\cos\theta, 2x\sin\theta\rangle \cdot \langle 0, 2\cos\theta, \sin\theta\rangle \, d\theta\, dx$$

$$= \int_0^1 \int_0^{2\pi} (2x\cos^2\theta + 2x\sin^2\theta) \, d\theta\, dx$$

$$= \int_0^1 \int_0^{2\pi} 2x \, d\theta\, dx = 4\pi \int_0^1 x\, dx = 2\pi x^2 \Big|_0^1 = 2\pi$$

$f(x,y,z)$ $g(x,y,z)$

20. Let C be the curve of intersection of the level surfaces $x^2 + y^2 = 169$ and $x^2 - z = 0$. Compute the parametric form of the tangent line to C at $(5, -12, 25)$.

(a) $\langle 24t - 5, 10t + 12, -240t - 25\rangle$

(b) $24(x-5) - 10(y+12) + 240(z-25) = 0$

(c) $\langle -24t + 5, -10t - 12, 240t + 25\rangle$

(d) $\langle 24t + 5, -10t - 12, 240t + 25\rangle$

(e) $\langle 24t + 5, 10t - 12, 240t + 25\rangle$

$\nabla f = \langle 2x, 2y, 0\rangle \quad \nabla g = \langle 2x, 0, -1\rangle$

$\nabla f(5,-12,25) = \langle 10, -24, 0\rangle \quad \nabla g(5,-12,25) = \langle 10, 0, -1\rangle$

$$\nabla f(5,-12,25) \times \nabla g(5,-12,25) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10 & -24 & 0 \\ 10 & 0 & -1 \end{vmatrix} = \langle 24, 10, 240\rangle$$

$\vec{r}(t) = \langle 5, -12, 25\rangle + t\langle 24, 10, 240\rangle = \langle 24t+5, 10t-12, 240t+25\rangle$