Math 20550 Calculus III Tutorial **Name:** ______________________________
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Tutorial Worksheet

Show all your work.
1. Consider the vector field $\mathbf{F}(x, y) = x^2\mathbf{i} + x\mathbf{j}$. Let C be a path counter-clockwise around the circle $x^2 + y^2 = 9$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. Is $\mathbf{F}$ conservative?

Solution: We can parametrize C by $\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j}$, $0 \le t \le 2\pi$. We compute $\mathbf{r}'(t) = -3\sin(t)\mathbf{i} + 3\cos(t)\mathbf{j}$. Therefore

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\ &= \int_0^{2\pi} (9\cos^2(t)\mathbf{i} + 3\cos(t)\mathbf{j}) \cdot (-3\sin(t)\mathbf{i} + 3\cos(t)\mathbf{j})\,dt \\ &= -27\int_0^{2\pi} \sin(t)\cos^2(t)\,dt + 9\int_0^{2\pi} \cos^2(t)\,dt.\end{aligned}$$

For the first integral, use $u = \cos(t)$.

$$\begin{aligned}-27\int_0^{2\pi} \sin(t)\cos^2(t)\,dt &= 27\int_0^{2\pi} u^2\,du \\ &= 9u^3\Big]_0^{2\pi} \\ &= 9\cos^3(t)\Big]_0^{2\pi} \\ &= 9 - 9 = 0.\end{aligned}$$

For the second integral, use the power-reducing formula.

$$\begin{aligned}9\int_0^{2\pi} \cos^2(t)\,dt &= \frac{9}{2}\int_0^{2\pi} (1 + \cos(2t))\,dt \\ &= \frac{9}{2}(t + \frac{1}{2}\sin(2t))\Big]_0^{2\pi} \\ &= \frac{9}{2}(2\pi - 0) = 9\pi.\end{aligned}$$

Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = 0 + 9\pi = 9\pi$. Because the integral of $\mathbf{F}$ is not zero along the closed curve C, $\mathbf{F}$ cannot be conservative (which can also be checked by taking the partials.)

2. Determine if the following vector field is conservative and find a potential function for the vector field if it is conservative.

$$\mathbf{F}(x, y) = (2xe^{xy} + x^2ye^{xy})\mathbf{i} + (x^3e^{xy} + 2y)\mathbf{j}$$

Solution: Let $P(x, y) = 2xe^{xy} + x^2ye^{xy}$ and $Q(x, y) = x^3e^{xy} + 2y$. First we compute

$$\frac{\partial P}{\partial y} = 2x^2e^{xy} + x^2e^{xy} + x^3ye^{xy} = 3x^2e^{xy} + x^3ye^{xy}.$$

$$\frac{\partial Q}{\partial x} = 3x^2e^{xy} + x^3ye^{xy}.$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, and $\mathbf{F}(x, y)$ is defined on the entire plane which is simply-connected, $\mathbf{F}$ is conservative.

Then we compute

$$f(x, y) = \int (x^3e^{xy} + 2y)\, dy = x^2e^{xy} + y^2 + h(x).$$

We compute

$$P(x, y) = \frac{\partial f}{\partial x} = 2xe^{xy} + x^2ye^{xy} + h'(x) = P(x, y) + h'(x).$$

This shows $h'(x) = 0$, so $h(x) = C$. Thus a potential function for $\mathbf{F}$ is

$$f(x, y) = x^2e^{xy} + y^2 + C.$$

3. A particle moves through a force field $\mathbf{F} = (2x + \sin(y))\mathbf{i} + (x\cos(y) - \sin(y))\mathbf{j}$ from $(0,0)$ to $(2,\pi)$. Show that $\mathbf{F}$ is conservative, and compute the work done by the force field ($W = \int_C \mathbf{F} \cdot d\mathbf{r}$).

Solution: Let $P(x,y) = 2x + \sin(y)$ and $Q(x,y) = x\cos(y) - \sin(y)$. We compute:

$$\frac{\partial P}{\partial y} = \cos(y) = \frac{\partial Q}{\partial x}.$$

Since $\mathbf{F}$ is defined on the entire plane which is simply-connected, it is therefore conservative.

Because we do not know the path the particle takes, we must use the fundamental theorem of line integrals. To do this, we need to find the potential function.

$$f(x,y) = \int (2x + \sin(y))\,dx = x^2 + x\sin(y) + h(y).$$

Now we take the y partial and compare:

$$Q(x,y) = \frac{\partial f}{\partial y} = x\cos(y) + h'(y).$$

Thus $h'(y) = -\sin(y)$, so $h(y) = \int(-\sin(y))\,dy = \cos(y) + C$. The choice of C does not matter, so let us set $C = 0$, giving

$$f(x,y) = x^2 + x\sin(y) + \cos(y).$$

By the fundamental theorem,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= f(2,\pi) - f(0,0) \\
&= (2^2 + 2\sin\pi + \cos\pi) - (0^2 + 0\sin 0 + \cos 0) \\
&= 3 - 1 = 2.
\end{aligned}$$

4. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ where $f(x, y, z) = \cos \pi x + \sin \pi y - xyz$ and C is any path that starts at $\left(1, \frac{1}{2}, 2\right)$ and ends at $(2, 1, -1)$.

Solution: By the Fundamental Theorem for line integrals,

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= f(2, 1, -1) - f\left(1, \frac{1}{2}, 2\right) \\ &= (\cos 2\pi + \sin \pi + 2) - \left(\cos \pi + \sin \frac{\pi}{2} - 1\right) \\ &= 3 - (-1) = 4.\end{aligned}$$