Math 20550 Calculus III Tutorial **Name:** ______________________________
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Tutorial Worksheet

1. Write the integral that corresponds to the surface area of the surface $x^2 + y^2 + z = 4$ above xy-plane.

(a ✗) $\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\; r\,dr\,d\theta$ (b) $\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\; dr\;d\theta$ (c) $\int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\; r\,dr\,d\theta$

(d) $\int_0^{2\pi}\int_0^2 \sqrt{1+r^2}\; r\,dr\,d\theta$ (e) $\int_0^{\pi}\int_0^2 \sqrt{1+4r^2}\; r\,dr\,d\theta$

Solution: Looking at the answers suggests using polar coordinates so $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $z = 4 - x^2 - y^2 = 4 - r^2$. Therefore

$$\mathbf{r}(r,\theta) = \left\langle r\cos(\theta), r\sin(\theta), 4 - r^2 \right\rangle$$

This surface is above the xy plane as long as $z \geqslant 0$ or $x^2 + y^2 \leqslant 4$ which is the disk of radius 2.

Then

$$\mathbf{r}_r(r,\theta) = \langle \cos(\theta), \sin(\theta), -2r\rangle$$
$$\mathbf{r}_\theta(r,\theta) = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \left\langle 2r^2\cos(\theta), -\left(-2r^2\sin(\theta)\right), r\cos^2(\theta) + r\sin^2(\theta)\right\rangle =$$

$$r\,\langle 2r\cos(\theta), 2r\sin(\theta), 1\rangle$$

Hence $|r\,\langle 2r\cos(\theta), 2r\sin(\theta), 1\rangle\,| = r\sqrt{4r^2+1}$.

$$\text{Surface Area} = \iint_T 1 dS = \iint_{Disk} r\sqrt{4r^2+1}\,dA = \int_0^{2\pi}\int_0^2 r\sqrt{4r^2+1}\,dr\,d\theta$$

2. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle y - \cos y, x\sin y\rangle$ and C is the circle $(x-3)^2+(y+4)^2 = 4$ oriented clockwise.

Solution: We use Green's Theorem to obtain

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_D (\partial_x Q - \partial_y P) dA = -\int\int_D dA,$$

where D is the region enclosed by our circle. Since the area of our circle is

$$A = 4\pi$$

and the circle is oriented clockwise, we have that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int\int_D dA = -(-4\pi) = 4\pi.$$

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle x^2 + y, 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region Dthat has area 6.

Solution: We use Green's Theorem to obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D (\partial_x Q - \partial_y P) dA = 2 \int\int_D dA.$$

Since the area of our region D is 6, we have that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 2 \int\int_D dA = 2 \cdot 6 = 12.$$

4. Find an equation for the tangent plane of the surface $z = x^2 + y^3 - 8$ at the point $(1, 2, 1)$.

(a) $x + 6y = 13$ (b̸) $-2x - 12y + z = -25$ (c) $x + 2y + z = 6$
(d) $2x + 8y + z = 19$ (e) $2x + 12y + z = 27$

Solution: Surface is $f(x, y, z) = x^2 + y^3 - z - 8 = 0$. $\nabla f = \langle 2x, 3y^2, -1 \rangle$. $\nabla f(1, 2, 1) = \langle 2, 12, -1 \rangle$ so

$$\langle 2, 12, -1 \rangle \bullet \langle x, y, z \rangle = \langle 2, 12, -1 \rangle \bullet \langle 1, 2, 1 \rangle$$

or

$$2x + 12y - z = 25$$

5. Determine two vectors that are tangent to the surface $\mathbf{r}(u, v) = \langle uv^2 - 2v, vu^2 - u, uv \rangle$ at the point $(0, 1, 2)$.

(a) $\langle 2, 4, 2 \rangle$, $\langle 1, 3, 1 \rangle$ (b̸) $\langle 2, 1, 1 \rangle$, $\langle 4, 3, 2 \rangle$ (c) $\langle 0, 1, 1 \rangle$, $\langle 4, 1, 1 \rangle$
(d) $\langle 1, 2, -1 \rangle$, $\langle 1, -2, 1 \rangle$ (e) $\langle 0, 2, -1 \rangle$, $\langle 1, -6, 3 \rangle$

Solution: We need to find u and v such that $\langle uv^2 - 2v, u^2 v - u, uv \rangle = \langle 0, 1, 2 \rangle$. Well, $uv = 2$ and also $u(uv - 1) = 1$ so $u = 1$ and $v = 2$. Then we need to compute

$$\begin{aligned} \mathbf{r}_u(u, v) &= \langle v^2, 2uv - 1, v \rangle & \mathbf{r}_u(1, 2) &= \langle 4, 3, 2 \rangle \\ \mathbf{r}_v(u, v) &= \langle 2uv - 2, u^2, u \rangle & \mathbf{r}_v(1, 2) &= \langle 2, 1, 1 \rangle \end{aligned}$$