Math 20550 Calculus III Tutorial **Name:** ______________________________
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Tutorial Worksheet

1. Calculate $\iint_S yx^2 dS$ where S is the surface $z = x + y^2$, $0 \leq x \leq 1$ and $0 \leq y \leq 2$.

Solution: Letting $\mathbf{r}(x, y) = xi + yj + (x + y^2)k$ we have

$$\mathbf{r}_x = i + k, \quad \mathbf{r}_y = j + 2yk$$

Therefore,

$$\mathbf{r}_x \times \mathbf{r}_y = -i - 2yj + k,$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1^2 + 4y^2 + 1}.$$

Therefore, using

$$\iint_S f(x, y, z)dS = \iint_D f(\mathbf{r}(x, y)|\mathbf{r}_x \times \mathbf{r}_y|dA,$$

we obtain

$$\iint_S yx^2 d\mathbf{S} = \int_0^1 \int_0^2 yx^2 \sqrt{1^2 + 4y^2 + 1} dA.$$

We separate the integrals

$$\iint_S yx^2 d\mathbf{S} = \sqrt{2} \int_0^1 x^2 dx \int_0^2 y\sqrt{1 + 2y^2} dy.$$

The first integral integrates to 1/3, and the second integral becomes

$$\int_0^2 y\sqrt{1 + 2y^2} dy = \frac{2}{12}(1 + 2y^2)^{3/2}|_0^2 = \frac{13}{3},$$

so we obtain

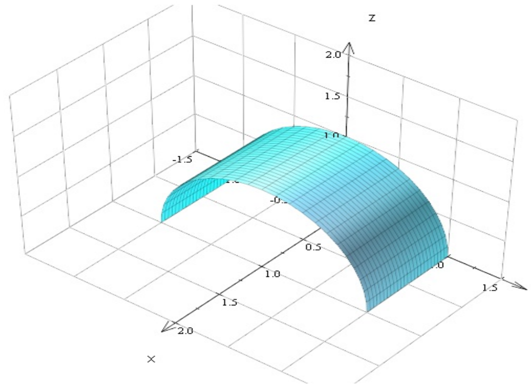
$$\iint_S yx^2 d\mathbf{S} = \frac{13\sqrt{2}}{9}.$$

2. Let S be the part of the cylinder $y^2 + z^2 = 1$, with $z \geqslant 0$, and $0 \leqslant x \leqslant 1$, and let S have the upward orientation. Determine which of the following equals $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = \langle 0, 0, z \rangle$.

(a) $\int_0^1 \int_{-1}^1 \sqrt{1 - y^2} dy\, dx$ (b) $\int_0^1 \int_{-1}^1 \sqrt{1 - x^2}\, dy\, dx$ (c) $\int_0^1 \int_{-1}^1 (1 - y^2)\, dy\, dx$

(d) $\int_0^1 \int_{-1}^1 (1 - x^2)\, dy\, dx$ (e) $\int_0^1 \int_{-1}^1 \left[\sqrt{1 - y^2}\right]^{-1} dy\, dx$

Solution: Our surface is:



This is the surface described by $z = \sqrt{1-y^2}$ where $0 \leqslant x \leqslant 1$, so letting x and y be our parameters, we have

$$\mathbf{X}(x, y) = \left\langle x, y, \sqrt{1-y^2} \right\rangle$$

where $0 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$. Let $D = \{(x, y) \,|\, 0 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}$. Now we need a normal vector field on S pointing upward. Begin by finding the two tangent vector fields X_x and X_y:

$$\mathbf{X}_x = \langle 1, 0, 0 \rangle \quad \text{and} \quad \mathbf{X}_y = \left\langle 0, 1, -\frac{y}{\sqrt{1-y^2}} \right\rangle$$

Then an upward normal vector field $\mathbf{n}$ is given by taking the cross product $\mathbf{X}_x \times \mathbf{X}_y$ We take the cross product in this order because of the right hand rule.

$$\mathbf{n} = \mathbf{X}_x \times \mathbf{X}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -\dfrac{y}{\sqrt{1-y^2}} \end{vmatrix} = \left\langle 0, \frac{y}{\sqrt{1-y^2}}, 1 \right\rangle$$

Finally,

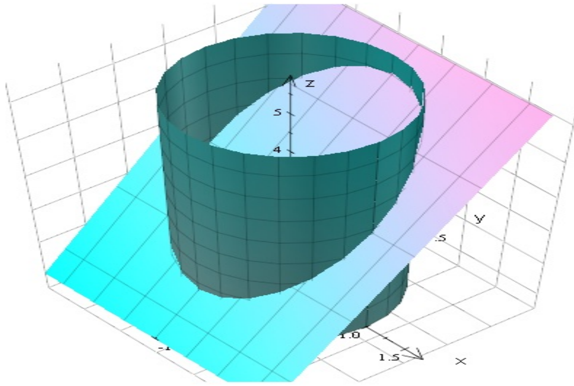
$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{X}(x, y)) \cdot \mathbf{n} \, dA \\ &= \iint_D \left\langle 0, 0, \sqrt{1-y^2} \right\rangle \cdot \left\langle 0, \frac{y}{\sqrt{1-y^2}}, 1 \right\rangle dA \\ &= \int_0^1 \int_{-1}^1 \sqrt{1-y^2} \, dy \, dx \end{aligned}$$

3. Let S be portion of the plane $z = 2y + 3$ inside the cylinder $x^2 + y^2 = 1$ oriented with the upward normal. Calculate $\iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S}$ where

$$\mathbf{F}(x, y, z) = \langle 2e^y - z, \cos(yz), xe^y \rangle$$

(a) 0 (b) 2π (c) $\dfrac{\pi}{\sqrt{5}}$ (d) $\sqrt{5}\pi$ (e) $\dfrac{2\pi}{\sqrt{5}}$

Solution: The region we have is:



We will use

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dS$$

Parameterising the plane that is inside the cylinder, we have

$$\mathbf{X}(x, y) = \langle x, y, 2y + 3 \rangle, \quad x^2 + y^2 \leqslant 1$$

and let $D = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$. First, finding the curl of $\mathbf{F}$:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2e^y - z & \cos(yz) & xe^y \end{vmatrix} = \langle xe^y + y\sin(yz), -(1 + e^y), -2e^y \rangle$$

The tangent vectors are:

$$\frac{\partial \mathbf{X}}{\partial x} = \langle 1, 0, 0 \rangle$$
$$\frac{\partial \mathbf{X}}{\partial y} = \langle 0, 1, 2 \rangle$$

By the right hand rule, the normal vector field is:

$$\mathbf{n} = \frac{\partial \mathbf{X}}{\partial x} \times \frac{\partial \mathbf{X}}{\partial y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \langle 0, -2, 1 \rangle$$

Notice we don't need to bother figuring out the x-component of $(\operatorname{curl} \mathbf{F})(\mathbf{X}(x, y))$ since it will be multiplied by 0 in the dot product. So

$$(\operatorname{curl} \mathbf{F})(\mathbf{X}(x, y)) = \langle \cdot, -(1 + e^y), -2e^y \rangle$$

and

$$(\operatorname{curl} \mathbf{F})(\mathbf{X}(r, \theta)) \cdot \mathbf{n} = (\cdot)0 + 2(1 + e^y) - 2e^y = 2$$

Finally, we have

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} = \iint_D 2\, dA = 2area(D) = 2(\pi \cdot 1^2) = 2\pi$$

4. Use Stokes' theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2 y\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise as viewed from above.

Solution: We calculate curl $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$. Stokes' theorem tells us that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S}.$$

Letting $P = x$, $Q = -y$, $R = 0$ and $g(x, y) = y^2 - x^2$, we may apply the formula

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_D (-Pg_x - Qg_y + R)\, dA$$

to obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D [-x(-2x) - (-y)(2y)]dA = 2\int_0^{2\pi}\int_0^1 r^2 r dr d\theta = \pi.$$