Math 20550 Calculus III Tutorial **Name:** ____________________
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Tutorial Worksheet

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1. The initial position and velocity of an object moving with acceleration $\mathbf{a} = e^t\mathbf{i}$ are $\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Find its position at time t.

Solution: To find velocity, we integrate the acceleration:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \left\langle e^t, 0, 0\right\rangle\,dt = \left\langle e^t, 0, 0\right\rangle + \mathbf{c}.$$

Now we solve for $\mathbf{c}$ using the initial velocity: $\mathbf{v}(0) = \langle 1, 1, 1\rangle = \langle e^0, 0, 0\rangle + \mathbf{c}$, so $\mathbf{c} = \langle 0, 1, 1\rangle$. Therefore $\mathbf{v}(t) = \langle e^t, 0, 0\rangle + \langle 0, 1, 1\rangle = \langle e^t, 1, 1\rangle$.

Then we integrate again to find position:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \left\langle e^t, 1, 1\right\rangle\,dt = \left\langle e^t, t, t\right\rangle + \mathbf{c}.$$

The initial position lets us solve for $\mathbf{c}$. Since $\mathbf{r}(0) = \langle 2, 3, 2\rangle = \langle e^0, 0, 0\rangle + \mathbf{c}$, we get $\mathbf{c} = \langle 1, 3, 2\rangle$, and so $\mathbf{r}(t) = \langle e^t, t, t\rangle + \langle 1, 3, 2\rangle$.

2. Find the equations for the normal and osculating planes to the curve $\mathbf{r}(t) = 2\cos(3t)\mathbf{i} + t\mathbf{j} + 2\sin(3t)\mathbf{k}$ at the point $(-2, \pi, 0)$.

Solution: Setting $\mathbf{r}(t_0) = \langle -2, \pi, 0\rangle$, we see from the $\mathbf{j}$ coordinate that $t_0 = \pi$. The normal plane is a plane perpendicular to the tangent vector $\mathbf{r}'(\pi)$ and containing the point $(-2, \pi, 0)$. We compute

$$\mathbf{r}'(t) = \langle -6\sin(3t), 1, 6\cos(3t)\rangle$$

giving $\mathbf{r}'(\pi) = \langle 0, 1, -6\rangle$. Therefore the equation of the normal plane is

$$\langle 0, 1, -6\rangle \bullet \langle x, y, z\rangle = \langle 0, 1, -6\rangle \bullet \langle -2, \pi, 0\rangle$$

which is $y - 6z = \pi$.

The osculating plane is a plane perpendicular to the binormal vector $\mathbf{B}(\pi)$ and containing the point $(-2, \pi, 0)$. The binormal is in the direction of $\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)$. We have

$$\mathbf{r}''(t) = \langle -18\cos(3t), 0, -18\sin(3t)\rangle$$

so $\mathbf{r}''(\pi) = \langle 18, 0, 0\rangle$. Then we get

$$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -6 \\ 18 & 0 & 0 \end{vmatrix} = \langle 0, -6 \cdot 18, -18\rangle = (-18)\,\langle 0, 6, 1\rangle$$

Using $\langle 0, 6, 1\rangle$ as the normal vector to the osculating plane, the equation is

$$\langle 0, 6, 1\rangle \bullet \langle x, y, z\rangle = \langle 0, 6, 1\rangle \bullet \langle -2, \pi, 0\rangle = 6\pi$$

or $6y + z = 6\pi$.

3. Find the unit tangent, unit normal, and binormal vectors to the curve $\mathbf{r}(t) = \langle t^2, t^3, t^4 + t^2 \rangle$ at $t = 1$.

Solution: First we compute the first and second derivatives:

$\mathbf{r}'(t) = \langle 2t, 3t^2, 4t^3 + 2t \rangle$ and $\mathbf{r}'(1) = \langle 2, 3, 6 \rangle$.

$\mathbf{r}''(t) = \langle 2, 6t, 12t^2 + 2 \rangle$ and $\mathbf{r}''(1) = \langle 2, 6, 14 \rangle$.

The unit tangent $\mathbf{T}(1)$ is a unit vector in the direction of $\mathbf{r}'(1)$, so

$$\mathbf{T}(1) = \frac{1}{|\langle 2, 3, 6 \rangle|} \langle 2, 3, 6 \rangle = \frac{1}{7} \langle 2, 3, 6 \rangle .$$

The binormal $\mathbf{B}(1)$ is a unit vector in the direction of $\mathbf{r}'(1) \times \mathbf{r}''(1)$.

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 6 \\ 2 & 6 & 14 \end{vmatrix} = \langle 6, -16, 6 \rangle .$$

$$\mathbf{B}(1) = \frac{1}{|\langle 6, -16, 6 \rangle|} \langle 6, -16, 6 \rangle = \frac{1}{\sqrt{82}} \langle 3, -8, 3 \rangle .$$

Finally, the unit normal vector is $\mathbf{N}(1) = \mathbf{B}(1) \times \mathbf{T}(1)$.

$$\mathbf{N}(1) = \frac{1}{7\sqrt{82}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -8 & 3 \\ 2 & 3 & 6 \end{vmatrix} = \frac{1}{7\sqrt{82}} \langle -57, -12, 25 \rangle .$$

4. Find the length of the curve $\mathbf{r}(t) = \langle e^{2t}, t, 2e^t \rangle$ on the interval $0 \le t \le 3$.

Solution: First compute the derivative:

$$\mathbf{r}'(t) = \left\langle 2e^{2t}, 1, 2e^t \right\rangle$$

$$|\,\mathbf{r}'(t)\,| = \sqrt{4e^{4t} + 4e^{2t} + 1} = \sqrt{(2e^{2t} + 1)^2} = 2e^{2t} + 1$$

(because $2e^{2t} + 1$ is always positive). Now the arc length formula gives

$$L = \int_0^3 |\,\mathbf{r}'(t)\,|\, dt = \int_0^3 (2e^{2t} + 1)\, dt = e^{2t} + t\Big]_0^3 = (e^6 + 3) - (1 + 0) = e^6 + 2.$$