Math 20550 Calculus III Tutorial **Name:** ____________________________
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Tutorial Worksheet

Show all your work.
1. Find the tangent plane and the normal line to the surface $x^2y + xz^2 = 2y^2z$ at the point $P(1, 1, 1)$.

Solution: The given surface can be expressed as a level surface by writing

$$f(x, y, z) = x^2y + xz^2 - 2y^2z = 0$$

which has the gradient

$$\nabla f(x, y, z) = \langle 2xy + z^2, x^2 - 4yz, 2xz - 2y^2 \rangle.$$

Thus, at the point $(1, 1, 1)$, we have $\nabla f(1, 1, 1) = \langle 3, -3, 0 \rangle$. This vector is the normal vector to the tangent plane and the direction vector for the normal line, so that the equation of the tangent plane at $(1, 1, 1)$ is

$$3(x - 1) - 3(y - 1) = 0 \;\Rightarrow\; x - y = 0,$$

and the equation for the normal line at $(1, 1, 1)$ is

$$r(t) = \langle 1, 1, 1 \rangle + t\langle 3, -3, 0 \rangle = \langle 1 + 3t, 1 - 3t, 1 \rangle.$$

2. Find the local maxima, minima, and saddle points of the function $z = x^3 + y^3 - 3xy + 1$.

Solution: Compute the gradient of $z = z(x, y)$, and then set it equal to zero to get:

$$\nabla z(x, y) = \langle 3x^2 - 3y, 3y^2 - 3x \rangle = \langle 0, 0 \rangle.$$

This gives the system of two equations:

$$\begin{aligned} y &= x^2 \\ x &= y^2 \end{aligned}$$

Any solution pair (x, y) of this system clearly must have non-negative x, y. The only solutions (critical points) are $(0, 0)$ and $(1, 1)$. At these points, we compute the Hessian of z:

$$\mathrm{Hess}_z(0, 0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}, \quad \mathrm{Hess}_z(1, 1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}.$$

Note that at $(0, 0)$, the determinant of the Hessian is **negative**, which means it is a saddle point. At $(1, 1)$, the determinant is **positive** and the **first entry of the first row is positive**, which means it is a local minimum.

3. Identify the maximum and minimum values attained by $z = x^2y - 2x^2$ within the triangle T bounded by the points $P(0,0)$, $Q(2,0)$, and $R(0,4)$.

Solution: First, we check for critical points in the interior of the triangle. The gradient of z is

$$\nabla z(x,y) = \langle 2xy - 4x, x^2 \rangle$$

But this only vanishes along the boundary of the triangle (where $x = 0$), so we move on to analyze the boundary.
From P to Q, the function z equals $-2x^2$, $0 \leq x \leq 2$, which has a maximum of 0 at 0 and a minimum of -8 at 2.
From P to R, the function z is identically zero.
From Q to R, $y = -2x + 4$, so the function z becomes

$$\begin{aligned} z &= x^2(-2x+4) - 2x^2 \\ &= -2x^3 + 2x^2, \text{ for } 0 \leq x \leq 2 \end{aligned}$$

Using Calc I tools, we check the derivative

$$\frac{dz}{dx} = -6x^2 + 4x = x(4 - 6x)$$

which is zero at $x = 0$ and $x = 2/3$. Checking these values, we see that z achieves a minimum of 0 at $x = 0$ and a maximum of $8/27$ at $x = 2/3$.
Comparing all these results, we conclude that on the whole triangle (including boundaries), the function reaches a global maximum of $8/27$ at $(2/3, 8/3)$ and a global min of -8 at $(2,0)$.

4. Identify the maximum and minimum values attained by $z = 4x^2 - y^2 + 1$ within the region R bounded by the curve $4x^2 + y^2 = 16$.

Solution: First, we check for critical points in the interior of the region. We have

$$\nabla z = \langle 8x, -2y \rangle$$

So the only critical point is $(0,0)$; but the Hessian at $(0,0)$ has negative determinant, hence $(0,0)$ is not a maximum, nor a minimum.
Now, we check for critical points along the boundary. The boundary consists of those points (x,y) such that $g(x,y) = 0$, where $g(x,y) = 4x^2 + y^2 - 16$. The method of Lagrange multipliers tells us that points along the boundary are critical when $\nabla z(x,y) = \lambda \nabla g(x,y)$. From this we get the system:

$$\begin{aligned} x &= \lambda x \\ y &= -\lambda y \\ 4x^2 + y^2 &= 16 \end{aligned}$$

We cannot have $\lambda = 0$, for otherwise $x = y = 0$, and this doesn't satisfy the restriction. Thus, either $x = 0$ (and $y \neq 0$) or the other way around, $y = 0$ and $x \neq 0$. It's easy to see that z has a maximum of 17 at $(-2,0)$ and $(2,0)$, and a minimum of -15 at $(0,4)$ and $(0,-4)$.